

Tetrational as special function

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Аннотация

Holomorphic tetrational (superexponential) to base e and its inverse function (arctetrational) are approximated with elementary functions.

1 Introduction

The tetrational and the superexponential refer to a holomorphic solution F of the equation

$$F(z+1) = \exp_b(F(z)). \quad (1)$$

Such equation is considered since years 1950 [1, 2, 3, 4, 5, 6, 7, 8, 9]; in particular, for the natural base $b=e$. Name “superexponential”, or “sexp”, indicates, that function F is *superfunction* [8, 10] of the exponential.

In general, for some function H , which can be called also the *transfer function* [10], a superfunction F is a holomorphic solution of the equation

$$F(z+1) = H(F(z)) . \quad (2)$$

Equation (1) is a special case of equation (2) for $H = \exp_b$. Then, multiplication is a superfunction of summation (addition of a constant), exponentiation is a superfunction of multiplication, and solution F of equation (1) is a superfunction of the exponential, id est a super-exponential.

The special case of a super-exponential, holomorphic at least in the right hand side of the complex plane, is called “tetrational”, $F = \text{tet}_b$, if it satisfies the additional condition

$$F(0) = 1 . \quad (3)$$

Four examples of tet_b are shown in figure 1 for $b=\sqrt{2}$, $b=\exp(1/e)$, $b=2$ and $b=e$.

The tetrational $\text{tet}_b(z)$ can be interpreted as result of exponentiation applied to unity z times, at least for integer values of z :

$$\text{tet}_b(z) = \underbrace{\exp_b \left(\exp_b \left(\dots \exp_b(1) \dots \right) \right)}_{z \text{ exponentiations}} \quad (4)$$

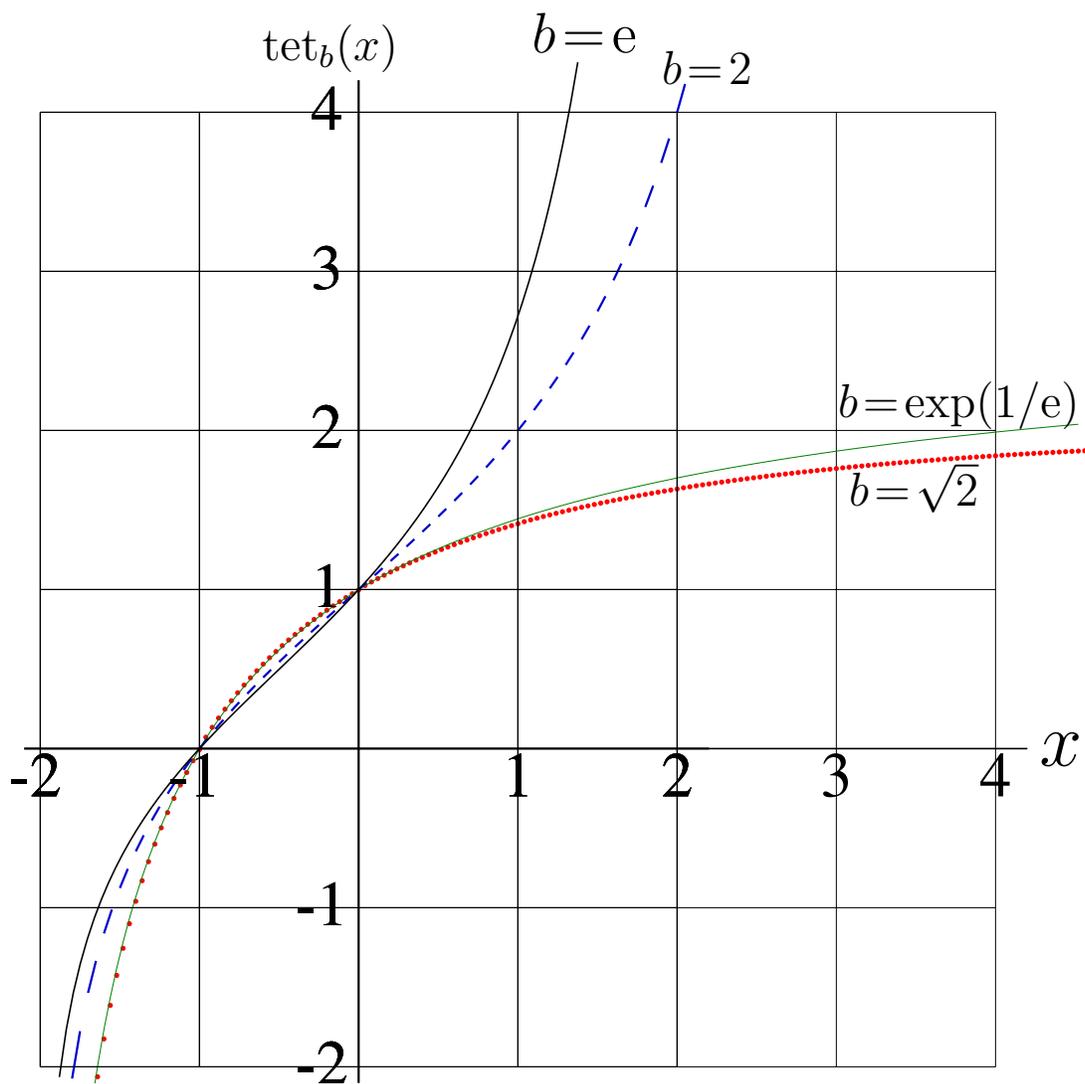


Рис. 1: Tetration $\text{tet}_b(x)$ at base $b = e$ (thick solid), $b = 2$ (dashed), $b = \exp(1/e)$ (thin solid) and $b = \sqrt{2}$ (dotted) versus real x .

The name “tetration” indicates, that this function is fourth in the sequence of functions (increment, addition, multiplication, exponential, tetration, pentation,...), where each element (except the zeroth element) is a superfunction for the previous element, and also the transfer function for the next element. The physical application of the superfunctions, that justify the term “transfer function” are suggested in [7, 10].

For complex values of the argument, the solution of equation (1) should be evaluated. The way of evaluation depends on b . At $1 < b \leq \exp(1/e)$, the regular iterations can be used, recovering the function through its Schröder function [2, 3, 4, 5, 6, 8]; for larger values through the Cauchy integral [7]. These representations were used to plot figure 1.

At base $b > \exp(1/e)$, the tetration can be expressed through the contour integral [7], assuming, that it is holomorphic on the domain

$$C = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq -2\} . \tag{5}$$

Such representation allows to express the derivative tet' and evaluate the inverse function, id est, arctetration $\text{ate} = \text{tet}^{-1}$. Also, the name “superlogarithm”, slog, is used for tet^{-1} , although arctetration is not a superfunction of the logarithm.

The arctetration ate satisfies the equation

$$\text{ate}(\exp(z)) = \text{ate}(z) + 1 \tag{6}$$

The uniqueness of the function ate , biholomorphic on the domain

$$G = \{z \in \mathbb{C} : \Re(z) \geq \Re(L); |z| < |L|\} \tag{7}$$

was shown [7, 9]. Here $L \approx 0.318 + 1.337i$ is a fixed point of logarithm, id est a solution of equation $L = \log(L)$. In the programming language Maple, this constant can be expressed as **conjugate(-LambertW(-1))**; while in notations of Mathematica it can be written as **Conjugate[-ProductLog[-1]]**.

Here, I consider the only one specific value $b = e$; $\log = \ln = \log_e$ and $H = \exp = \exp_e$. However, tetration for other values of base $b > e^{1/e}$ can be treated in similar way. Below, I suggest the approximation for the function $\text{tet} = \text{tet}_e$ shown in figure 1 with thick solid line; but this approximation is not limited to real values of the argument.

2 Properties of tet and ate

Any efficient approximation of any function should take into account its asymptotic properties. This section summarizes the basic knowledge about functions $f = \text{tet}(z)$ and $f = \text{ate}(z)$, that follow from the representation through the Cauchy integral [7]. Behavior of functions $f = \text{tet}(z)$ and $f = \text{ate}(z)$ in the complex plane is shown in figure 2 with lines $p = \Re(f) = \text{const}$ and $q = \Im(f) = \text{const}$. Levels of integer values p and q are shown with thick dark lines. Thick light lines indicate the levels $p = \Re(L)$ and $|q| = \Im(L)$. The thin lines indicate the intermediate levels. The shaded sickle indicate the set G by (7). The upper tip of the sickle is L ; the lower tip is L^* . The shaded strip shows the domain $\text{ate}(G)$.

Function tet has the branch point -2 ; the cut is directed along the real axis to $-\infty$. Position of the cut line, from -2 to $-\infty$ on the real axis, is determined by the condition $\text{tet}(z^*) = \text{tet}(z)^*$.

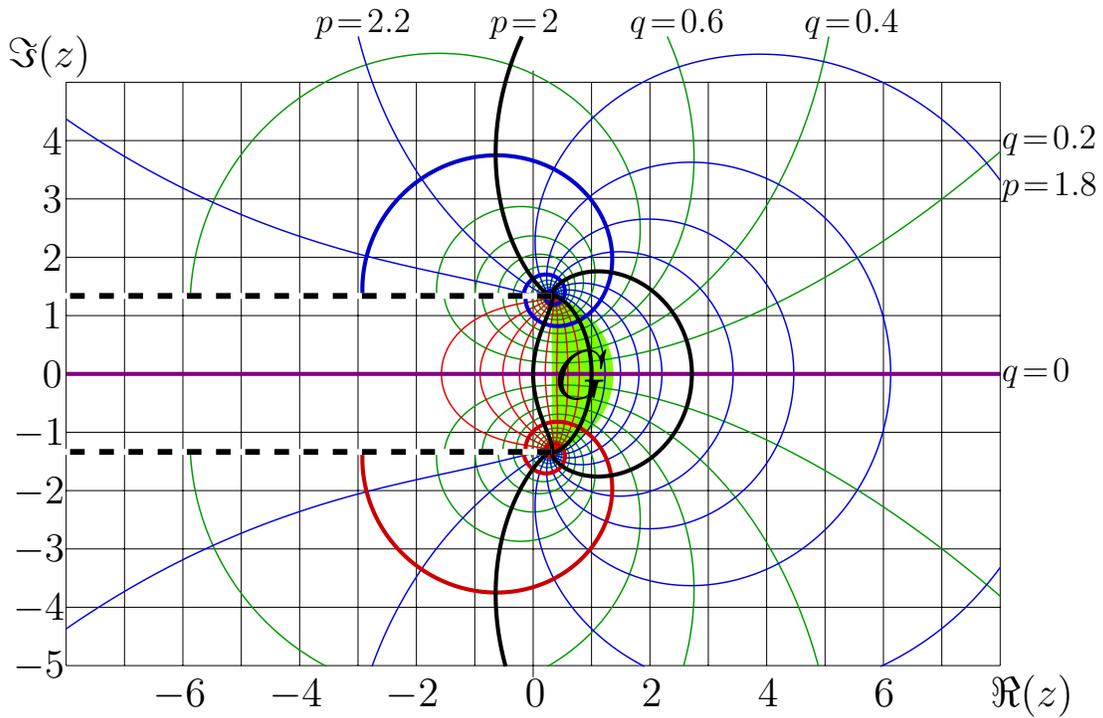
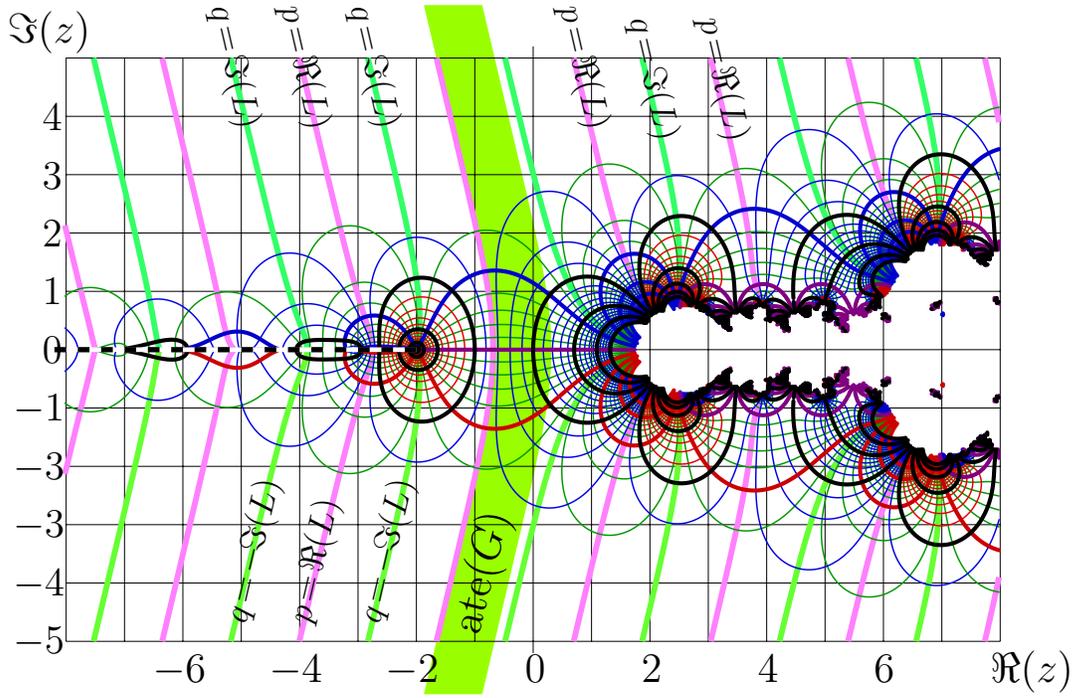


Рис. 2: functions $f = \text{tet}(z)$ and $f = \text{ate}(z)$ in the complex z -plane. Lines show levels $p = \Re(f) = \text{const}$ and $q = \Im(f) = \text{const}$.

Function ate has two branch points, L and L^* , and for the implementation we need to choose the cut lines. In the previous article [7], the cut lines run along the level $\Re(\text{tet}(z)) = -2$ (See figure 8 in [7];) these cuts wind around the branch points, and the calculation of the cut line slows down the algorithm of the evaluation of function ate . Therefore, in this paper, the cutlines are placed horizontally.

Function tet asymptotically approach its limiting values L in the upper halfplane and L^* in the lower halfplane. This approaching is seen in the figure 2 in the region, where the lines $p = \Re(L)$ look parallel to the lines $q = \Im(L)$. The approach to value L is exponential [7]. The approximation of $\text{tet}(z)$ at large values of $\Im(z)$ should use this property. In the left hand side of the complex plane and also in vicinity of the real axis function $f = \text{tet}$ satisfies not only equation (1), but also the “inverse” equation, id est

$$\log(f(z+1)) = f(z) , \quad \forall z \in C : |\Im(f(z))| < \pi. \quad (8)$$

Equations (1) and (8) simplify the fitting of the function. For the implementation of tetration, it is sufficient to approximate it in some domain in the complex z -plane, that extends from $-\infty$ to $i\infty$ in such a way that its overlap with set $\Im(z) = \text{const}$ is not shorter than unity. This domain may partially overlap with the image $\text{tet}(G)$ of domain G . In particular, such a domain can be a strip $|\Re(z)| \leq 1/2$, used in [7]; this region can be also an “alternative strip” $-1 \leq \Re(z) \leq 0$, suggested for the independent verification of that result. In a similar way, for the implementation of function ate , it is sufficient to approximate it in some domain that extends from L^* to L in such a way that the exponential of the left margin belongs to the domain. Such approximations below were calculated using the discrete representation of the Cauchy contour integral [7].

3 Implementation of tetration: fima

In order to distinguish functions tet and ate from their approximations, I give a specific name to each of them. The approximation of tet at large values of its argument can be build up using the asymptotic representation

$$\text{tet}(z) = L + \sum_{n,m} \mathcal{A}_{m,n} \exp(Lnz + \alpha mz) \quad (9)$$

The substitution into equation (1) and treating $\exp(Lz)$ as small parameter gives $\alpha = 2\pi i$, and equations for the coefficients \mathcal{A} .

Taking into account only few of terms in expansion (9) gives the asymptotic approximation, I call it fima (Fast approximation at large IMaginary part of the Argument):

$$\text{fima}(z) = \sum_{n=0}^N a_n \varepsilon^n + \beta \varepsilon \exp(2\pi i z) , \quad (10)$$

where the small parameter is

$$\varepsilon = \exp(Lz + R) , \quad (11)$$

and the coefficients

$$a_0 = L \approx 0.3181315052047641353 + 1.3372357014306894089 i \quad (12)$$

$$a_1 = 1 \quad (13)$$

$$a_2 = \frac{1/2}{L-1} \approx -0.1513148971556517359 - 0.2967488367322413067 i \quad (14)$$

$$a_3 = \frac{a_2 + 1/6}{L^2 - 1} = \frac{2 + L}{6(L-1)(L^2-1)} \approx -0.03697630940906762 + 0.09873054431149697 i \quad (15)$$

$$a_4 = \frac{6 + 6L + 5L^2 + L^3}{24(L-1)^3(L+1)(L^2+L+1)} \approx 0.0258115979731401398 - 0.017386962126530755 i \quad (16)$$

$$a_5 = \frac{24 + 36L + 46L^2 + 40L^3 + 24L^4 + 9L^5 + L^6}{120(L-1)^4(L+1)^2(1+L+2L^2+L^3+L^4)} \approx -0.0079444196 + 0.00057925018 i \quad (17)$$

Parameter R is introduced in order to set $a_1 = 1$ and keep simple expressions of other coefficients a through the fixed point L of the logarithm. Parameter R can be defined as a complex number such that, at fixed values of $\Re(z)$ and large values $\Im(z) \gg 1$,

$$\text{tet}(z) = L + \exp(LZ + R) + \mathcal{O}(\exp(2Lz)) . \quad (18)$$

The increase of number of terms in the polynomial (10) and addition of polynomials with factors $\exp(2\pi iz)$, $\exp(4\pi iz)$, etc. improves the approximation, but for the prototype of the complex < double > numerical implementation, constructed below, it is sufficient to take 6 terms in the sum (10), setting $bN=5$, and only the single term proportional to $\exp(2\pi iz)$.

I approximate parameters R and β , fitting the numerical solution by [7]:

$$R \approx 1.0779614375280 - 0.94654096394782 i \quad (19)$$

$$\beta \approx 0.12233176 - 0.02366108 i . \quad (20)$$

These values are expected to approximate the fundamental mathematical constants. Approximation fima by (10) is plotted in the top picture of the figure 3 in the same notations as in figure 2. The bottom picture shows the agreement function

$$D_0 = -\lg \left| \exp(\text{fima}(z+1)) - \text{fima}(z) \right| . \quad (21)$$

This function characterizes the residual at the substitution $F \rightarrow \text{fima}$ into equation (1). The level $D=1$ is shown with very thick light line at the bottom; the levels $D=2, 4, 6$ are shown with thin black lines; The levels $D=10, 12, 14$ are shown with thick lines; Roughly, the agreement function indicates, how many correct decimal digits may one get with this approximation. In particular, above the drawn lines, with this approximation, one may expect to get at least 14 correct decimal digits, but for values below the lowest thickest line, even the first digit of this approximation is doubtful.

With the conjugated approximation $\text{fima}(z^*)^*$, the fit by (10) covers a big part of the complex z -plane, but it is not good for small values of the imaginary part of the argument.

4 Approximation of sexp : expansion at zero

The Taylor series has radius of convergence, equal to the distance from the expansion point to the nearest singularity. In the case of tetration, the truncated sum of the MacLaurin series

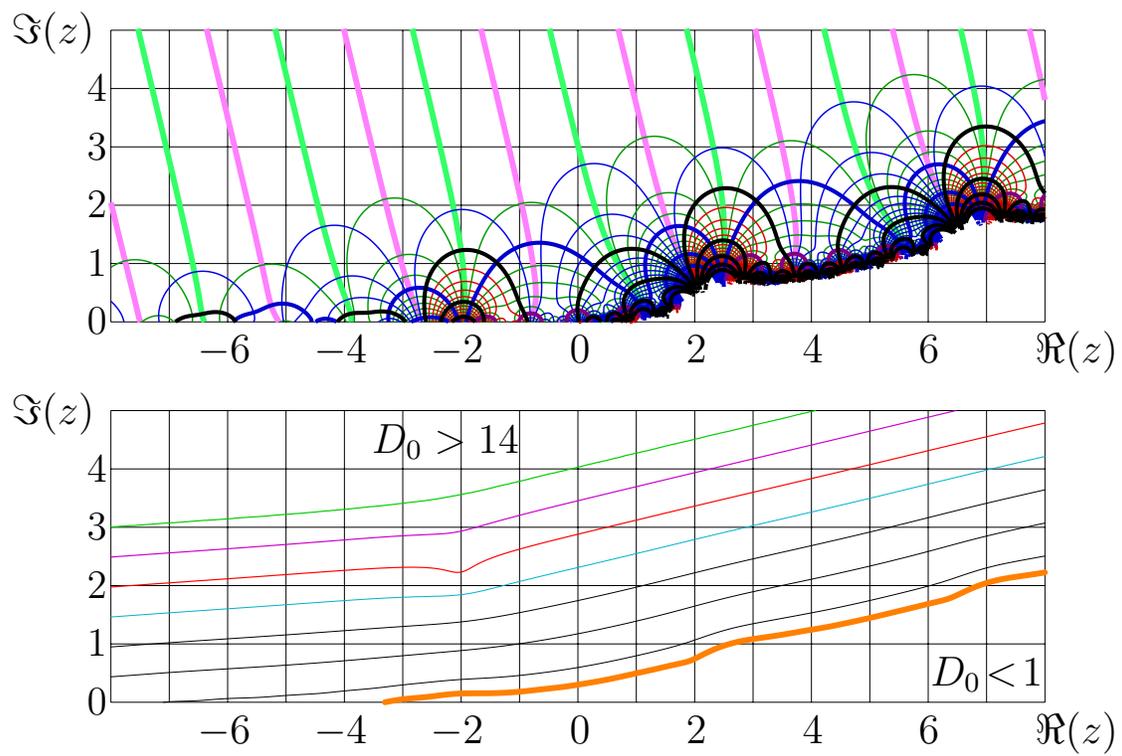


Рис. 3: Approximation $f = \text{fima}(z)$ by (10) and agreement D_0 by (21) in the complex z -plane.

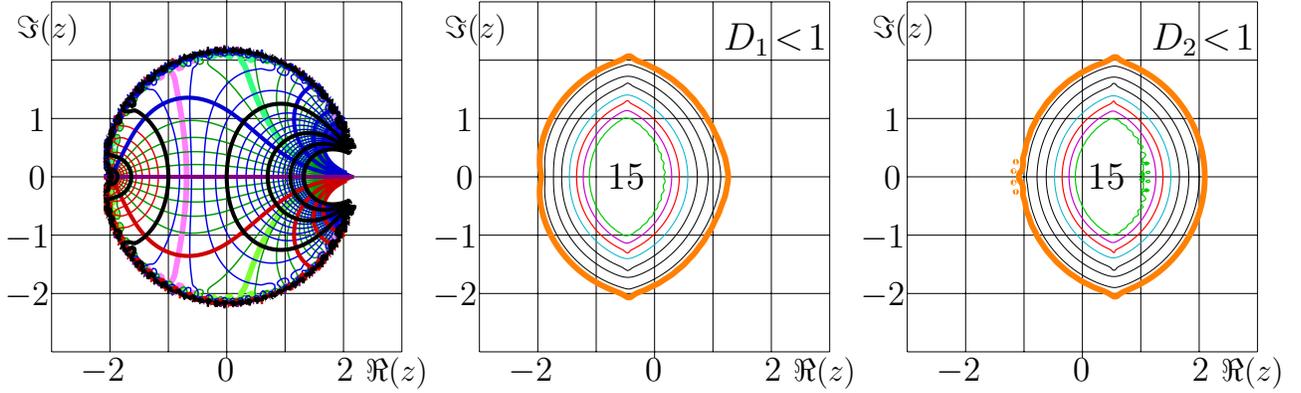


Рис. 4: Approximation $f = \text{naiv}(z)$ of tet with the truncated Taylor expansion (22) at zero, the left picture; the agreements D_1 and D_2 by (23) and (24), the central and the right ones.

gives the approximation

$$\text{naiv}(z) = \sum_{n=0}^{N-1} c_n z^n, \quad \text{tet}(z) = \text{naiv}(z) + \mathcal{O}(z^N) \quad (22)$$

usable at $|z| < 2$. The approximation with $N = 10$ is shown in the figure 4; is generated using the C++ source from CZ [11]. Approximations for the coefficients c_n are shown in the first column of Table 1. The 0th column of table 1 indicates the number n of the coefficient; the first column indicates the value of the coefficient c_n in equation (22).

The precision of the approximation (22) can be characterized with the agreement can be estimated with the agreement functions

$$D_1 = -\lg |\exp(\text{naiv}(z-1)) - \text{naiv}(z)| \quad \text{and} \quad (23)$$

$$D_2 = -\lg |\log(\text{naiv}(z+1)) - \text{naiv}(z)| \quad (24)$$

these agreements are plotted in the central and the right hand side pictures in figure 4. The digits “15” indicate the region, where the agreement is larger than 14. The figure indicates that at $|z| \leq 1$, the truncated Taylor series gives of order of 15 significant figures

In order to extend the range of approximation, it worth to “switch out” the nearest logarithmic singularity at -2 , expanding the function $\text{tet}(z) - \log(z+2)$ instead of $\text{tet}(z)$; let

$$\text{maclo}(z) = \log(z+2) + \sum_{n=0}^{N-1} s_n z^n; \quad (25)$$

$$\text{tet}(z) = \text{maclo}(z) + \mathcal{O}(z^N). \quad (26)$$

The name maclo (MAClaurin expansion with LOrarithm) indicates, that the tetration with subtracted logarithm is approximated with the truncated MAClaurin series. The first coefficients s of this expansion are evaluated in the second column in the table 1.

Function maclo is shown in the figure 5 for $N = 101$. The generator [12] is used to plot the figure. The range of approximation of tet with function maclo is significantly wider, than that by the Taylor expansion of tet at zero. The right hand side of the same figure 5, shows also the

Table 1. Coefficients of the series (22), (25) and (29).

n	c_n	s_n	$\Re(t_n)$	$\Im(t_n)$
0	1.000000000000000	0.30685281944005	0.37090658903229	1.33682167078891
1	1.09176735125832	0.59176735125832	0.01830048268799	0.06961107694975
2	0.27148321290170	0.39648321290170	-0.04222107960160	0.02429633404907
3	0.21245324817626	0.17078658150959	-0.01585164381085	-0.01478953595879
4	0.06954037613999	0.08516537613999	0.00264738081895	-0.00657558130520
5	0.04429195209047	0.03804195209047	0.00182759574799	-0.00025319516391
6	0.01473674209639	0.01734090876306	0.00036562994770	0.00028246515810
7	0.00866878181723	0.00755271038865	0.00002689538943	0.00014180498091
8	0.00279647939839	0.00328476064839	-0.00003139436775	0.00003583704949
9	0.00161063129058	0.00139361740170	-0.00001376358453	-0.00000183512708
10	0.00048992723148	0.00058758348148	-0.00000180290980	-0.00000314787679
11	0.00028818107115	0.00024379186661	0.00000026398870	-0.00000092613311
12	0.00008009461254	0.00010043966462	0.00000024961828	-0.00000013664223
13	0.00005029114179	0.00004090111776	0.00000007899707	0.00000003171468
14	0.00001218379034	0.00001654344436	0.00000000637479	0.00000002270476
15	0.00000866553367	0.00000663102846	-0.00000000341142	0.00000000512289
16	0.00000168778232	0.00000264145664	-0.00000000162203	0.00000000031619
17	0.00000149325325	0.00000104446533	-0.00000000038743	-0.00000000027282
18	0.00000019876076	0.00000041068839	-0.00000000001201	-0.00000000013440
19	0.00000026086736	0.00000016048059	0.00000000002570	-0.00000000002543
20	0.00000001470995	0.00000006239367	0.00000000000935	0.00000000000045
21	0.00000004683450	0.00000002412797	0.00000000000170	0.000000000000186
22	-0.00000000154924	0.00000000928797	-0.00000000000005	0.000000000000071
23	0.00000000874151	0.00000000355850	-0.00000000000016	0.000000000000012
24	-0.00000000112579	0.00000000135774	-0.00000000000005	-0.000000000000001
25	0.00000000170796	0.00000000051587	-0.00000000000001	-0.000000000000001

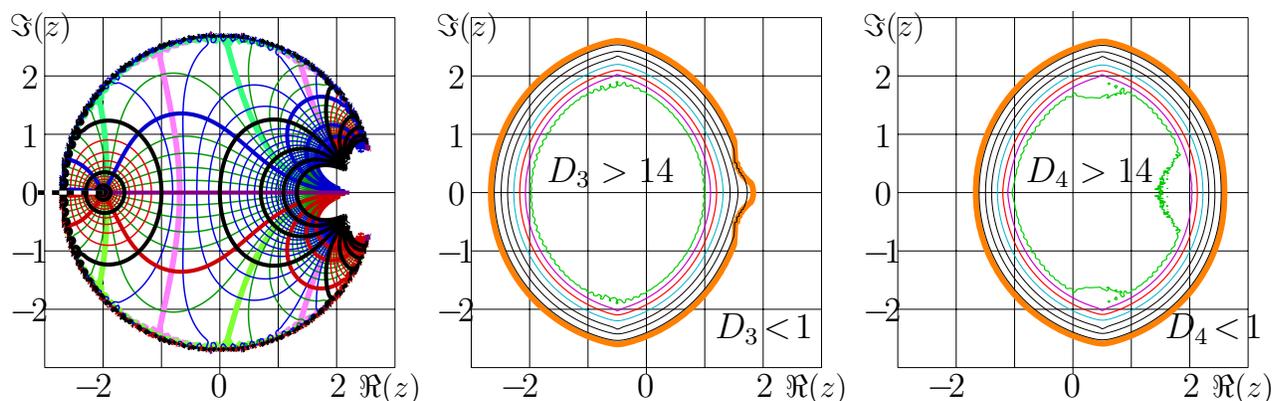


Рис. 5: function $f = \text{maclo}(z)$ by (25) at $N = 101$ in the complex z -plane, left; Agreements D_3 and D_4 by (27) and (28), center and right.

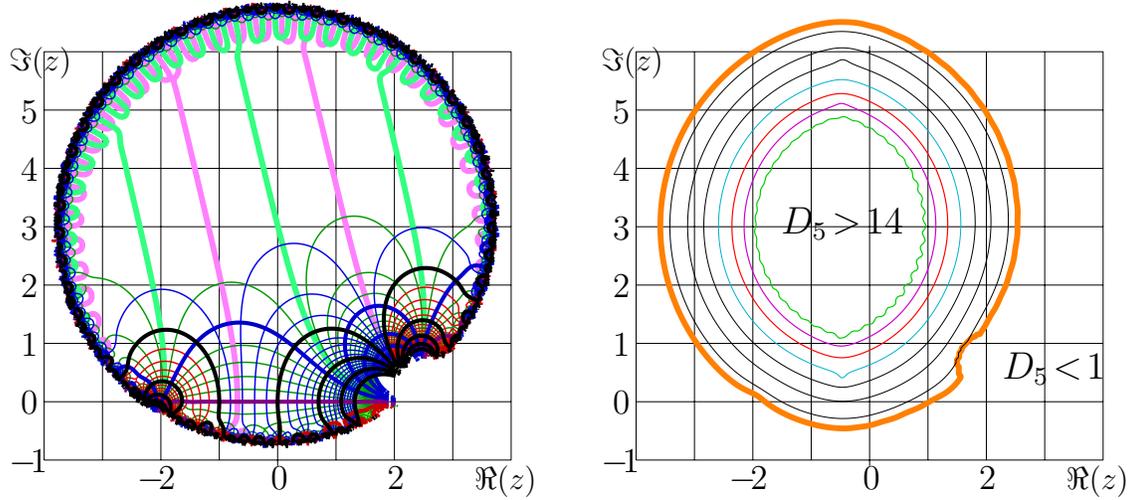


Рис. 6: Function $f = \text{tai}(z)$ by (29) for $N = 51$ in the complex z -plane and agreement D_5 by (30).

agreements

$$D_3 = -\lg \left| \exp(\text{maclo}(z+1)) - \text{maclo}(z) \right| \quad (27)$$

$$D_4 = -\lg \left| \log(\text{maclo}(z-1)) - \text{maclo}(z) \right| \quad (28)$$

Within the central loop, the residuals at the substitution $\text{tet} \rightarrow \text{maclo}$ into equations (1), (8) are of order of 10^{-15} .

5 Approximation of tet : Taylor expansion at $3i$.

The plots of the agreement functions D in figures 3 and 5 indicate, that in the intermediate range $z \approx 3i$, each of approximations $\text{fima}(z)$ and $\text{maclo}(z)$ return only few correct significant figures. For this reason I suggest the straightforward Taylor expansion at the intermediate point $z = i$. I call it “tai” (TAYlor expansion centered at at the Imaginary axis):

$$\text{tai}(z) = \sum_{n=0}^{N-1} t_n (z-3)^n \quad (29)$$

The coefficients of this series are evaluated in the last columns in the Table 1. For $N = 51$, function tai is shown 6. The figure is plotted with generator [13]. The precision of the approximation of the solution of equations (1), (8) is characterized with agreement

$$D_5 = -\lg \left| \log(\text{tai}(z+1)) - \text{tai}(z) \right| . \quad (30)$$

This function is plotted in the right hand side of figure 6.

The mutual agreement of the approximations above can be characterized with functions

$$D_6 = -\lg \left| \text{fima}(z) - \text{tai}(z) \right| \quad (31)$$

$$D_7 = -\lg \left| \text{maclo}(z) - \text{tai}(z) \right| \quad (32)$$

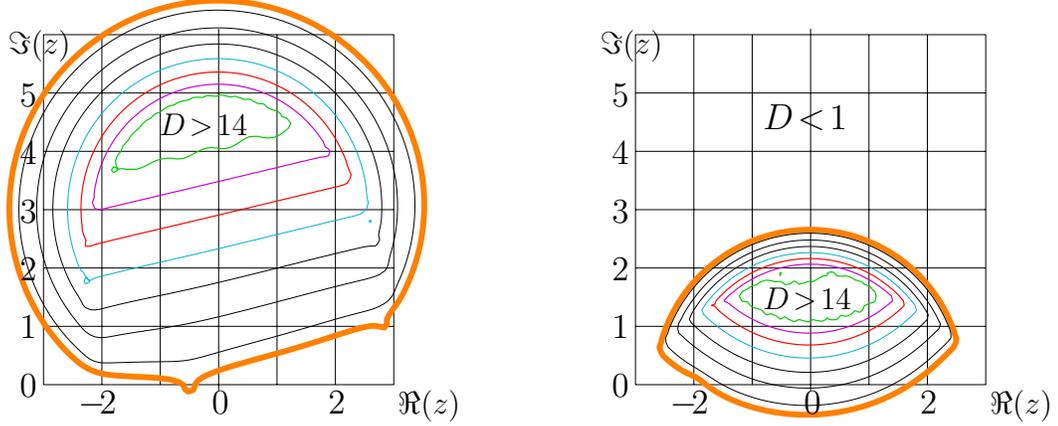


Рис. 7: Comparison of approximations tai by (29) to fima by (10) and to maclo by (25); agreements $D=D_6$ and $D=D_7$ by (31),(32) in the complex z -plane.

These functions are shown in figure 7. Within the inner loops in the pictures of the figure 7, the modulus of the difference between the approximations does not exceed 10^{-14} . On the base of figure 7, I suggest the following approximation:

$$fse(z) = \begin{cases} fima(z) & , & 4.5 < \Im(z) \\ tai(z) & , & 1.5 < \Im(z) \leq 4.5 \\ maclo(z) & , & -1.5 \leq \Im(z) \leq 1.5 \\ tai(z^*)^* & , & -4.5 \leq \Im(z) < -1.5 \\ fima(z^*)^* & , & \Im(z) < -4.5 \end{cases} \quad (33)$$

This approximation can be compared to previous results. Below, I analyze the deviation $fse(z) - F_4(z)$, where $F_4(z)$ is approximation, obtained by the straightforward implementation of the contour integral [7]. The left hand picture of figure 8 shows the agreement

$$D_8 = -\lg |fse(z) - F_4(z)| \quad (34)$$

of approximation fse with the approximation F_4 obtained through the direct implementation of the contour integral [7]. Figure 8 reveals the defects of each approximation. The jumps at $\Im(z) = 1.5$ and at $\Im(z) = 2.5$ should be attributed to the transition from function maclo to function tai and from function tai to function fima in the combination FSE. Jumps at half-integer values of $\Re(z)$ should be attributed to the discontinuities of function F_4 , which extends the approximation with the contour integral, valid for $|\Re(z)| < 1$, from the interval $|\Re(z)| \leq 1/2$. The rounding errors appear as irregular dots. Within the strip $|\Re(z)| < 1.5$, the irregularities of all three approximations are of order of 10^{-14} .

In the right hand side of figure 8 the similar agreement is shown for function F_5 , which is analogy of function F_4 , but the base strip is displaced for $-1/2$. Approximation F_5 has jumps at integer values of the real part of the argument. These jumps are also small; the agreement is at least not worse than for function F_4 .

In such a way, the deviation of all the approximations we count for today is of order of 10^{-14} . On the base of figures 7, 8, I suggest the final approximation FSE (Fast Super Exponential) of

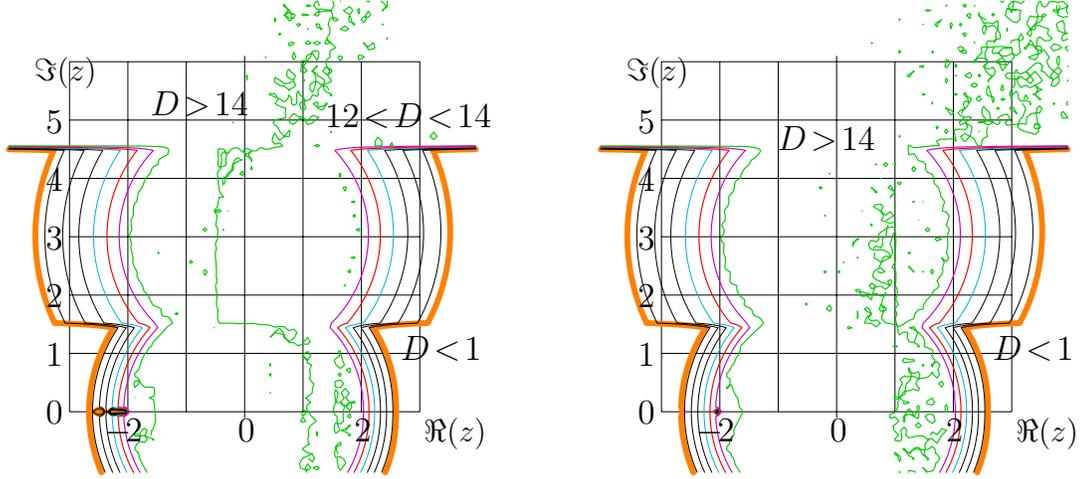


Рис. 8: Agreement $D = D_8$ by (34), left; the similar agreement for the contour integral with base domain shifted for -0.5 .

tetrational tet:

$$\text{FSE}(z) = \begin{cases} \text{FIMA}(z) & , \quad 4.5 < \Im(z) \\ \text{TAI}(z) & , \quad 1.5 < \Im(z) \leq 4.5 \\ \text{MACLO}(z) & , \quad -1.5 \leq \Im(z) \leq 1.5 \\ \text{TAI}(z^*)^* & , \quad -4.5 \leq \Im(z) < -1.5 \\ \text{FIMA}(z^*)^* & , \quad \Im(z) < -4.5 \end{cases} \quad (35)$$

where

$$\text{FIMA} = \begin{cases} \text{fima}(z) & , \quad \Im(z) > 4 + 0.2379 \Re(z) \\ \exp(\text{FIMA}(z-1)) & , \quad \Im(z) \leq 4 + 0.2379 \Re(z) \end{cases} \quad (36)$$

$$\text{TAI} = \begin{cases} \text{tai}(z) & , \quad |\Re(z)| \leq 0.5 \\ \log(\text{TAI}(z+1)) & , \quad \Re(z) < -0.5 \\ \exp(\text{TAI}(z-1)) & , \quad \Re(z) > 0.5 \end{cases} \quad (37)$$

$$\text{MACLO} = \begin{cases} \text{tai}(z) & , \quad |\Re(z)| \leq 0.5 \\ \log(\text{MACLO}(z+1)) & , \quad \Re(z) < -0.5 \\ \exp(\text{MACLO}(z-1)) & , \quad \Re(z) > 0.5 \end{cases} \quad (38)$$

This approximation provides of order of 14 correct significant figures of the holomorphic tetration tet and agrees with the previous results [7].

Up to my knowledge, the function FSE is the most precise and the fastest among ever reported approximations of the tetrational. Many terms are kept in the approximations (29) and (25) in order to provide the wide range of the overlapping in figures 7 and 8. At the final step of the implementation, the number of terms can be reduced, boosting the algorithm without loss of the precision. Especially this applies to the evaluation of tetration along the real axis: it is sufficient to have a good approximation of $\text{tet}(z)$ for $|z| \leq 1/2$, which is only a quarter of the radius of the precise approximation with function `maclo`.

By requests of colleagues the algorithm is translated from language C++ into language Mathematica [14]. As verification of this algorithm, the first, second, third and fourth derivatives

of tetrational tet are plotted in figure 9 as functions of real argument; however, the algorithm evaluates also the tetration and its derivatives of complex argument.

The good overlapping of the ranges of approximation of tetration tet by various algorithms confirm their validity. Perhaps, some increase of the relative error may take place at the sequential application of exp required at large values of the real part of the argument. At moderate values of the real part of the argument, all the 3 approximations FSL by (35), and F_4 by [7] and its modification F_5 (see figure 8) seem to have comparable errors at the level of 10^{-14} .

6 Implementation of arctetration: function FSL

The inverse function of tetration, id est, arctetration $\text{ate} = \text{tet}^{-1}$, satisfies the equations

$$\text{ate}(z) = \text{ate}(\exp(z)) - 1 \quad (39)$$

$$\text{ate}(z) = \text{ate}(\log(z)) + 1 \quad (40)$$

at least for $z \in G$; and $\text{ate}(1) = 0$. In principle, function ate can be implemented as numerical solution of equation $\text{tet}(\text{ate}(z)) = z$; however, such implementation is much slower than the approximation with appropriate elementary functions.

The first (and naive) attempt to approximate function ate is, of course, the Taylor expansion at unity. The coefficients of such an expansion can be found, inverting the powerseries naiv by (22). The radius of convergence of the resulting expansion is $|L| \approx 1.5$; and the approximation is especially poor in vicinity of the fixed points L and L^* of logarithm.

The better approach is to expand the function

$$\text{ate}(z) - \frac{\log(z-L)}{L} - \frac{\log(z-L^*)}{L^*} \quad (41)$$

at $z=1$. Such an expansion leads to the approximation

$$\text{fsl}(z) = \frac{\log(z-L)}{L} + \frac{\log(z-L^*)}{L^*} + \sum_{n=0}^{N-1} u_n (z-1)^n ; \quad (42)$$

$$\text{ate}(z) = \text{fsl}(z) + \mathcal{O}((z-1)^N) \quad (43)$$

The approximations of first coefficients of such an expansion are shown in table 2. The approximation fsl at $N=70$ is shown in figure 10.

Formally, the Taylor series of function (41) developed at $z=1$ has the same radius of convergence as the direct Taylor expansion of function ate. However, practically, at the numerical implementation, the convergence for representation (42) is much faster, than for the straightforward Taylor expansion. Function fsl approximates function ate even at the edge of the range of convergence, and, in particular, in vicinity of the tips of the sickle G by (7), id est, in vicinity of points L and L^* . Function fsl has the same branchpoints L and L^* , as function ate, and also is infinite in these points.

In order to characterize the residuals at the substitution $\text{ate} \rightarrow \text{fsl}$ into equations (39), (40), the agreements

$$D_A = -\lg \left| \text{fsl}(\exp(z)) - 1 - \text{fsl}(z) \right| \quad (44)$$

$$D_B = -\lg \left| \text{fsl}(\log(z)) + 1 - \text{fsl}(z) \right| \quad (45)$$

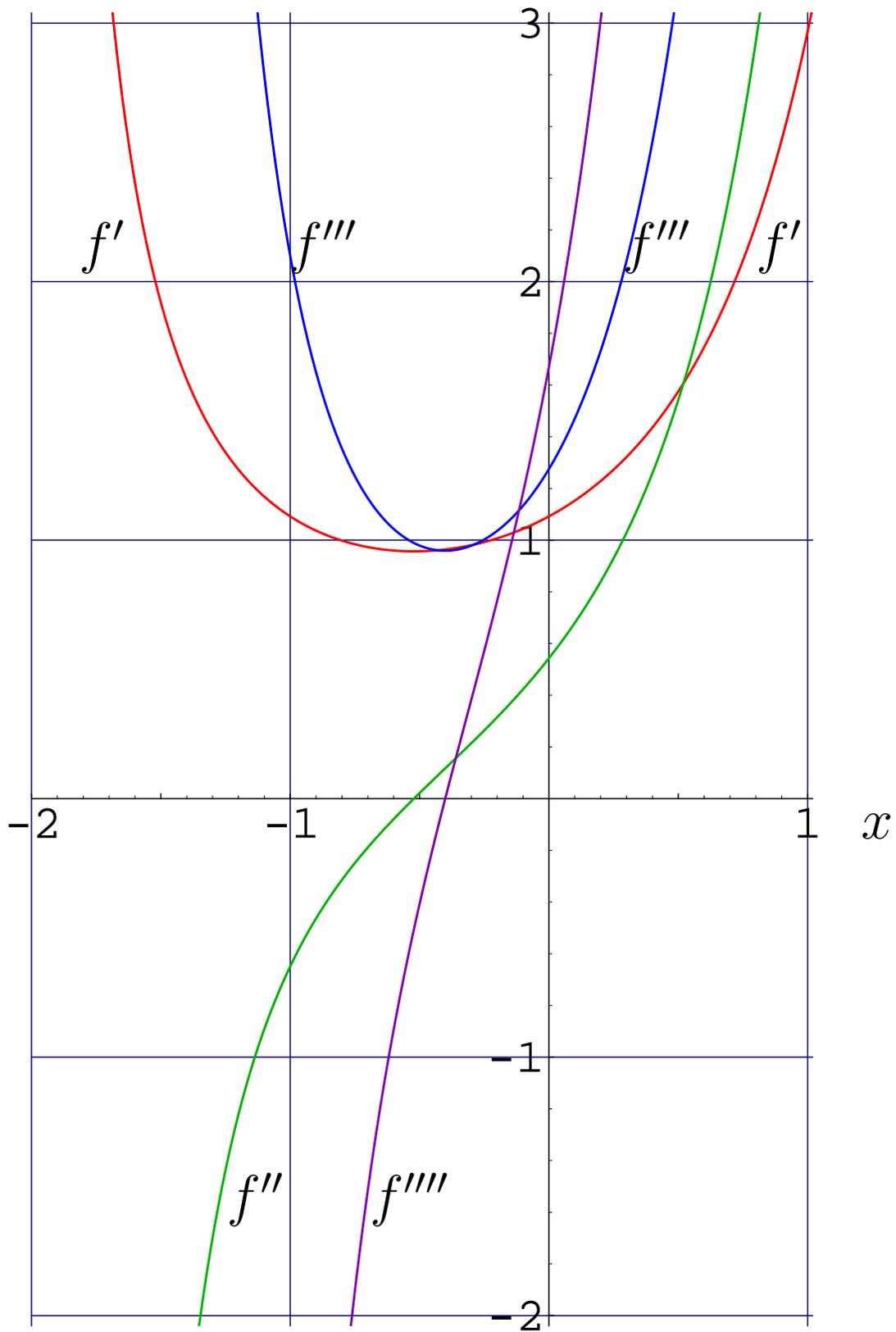


Рис. 9: First four derivatives of $f = \text{tet}(x)$ as functions of real x .

Table 2. Coefficients u_n in expansion (42).

n	u_n	n	u_n	n	u_n
0	1.41922521550451	10	0.00000003111805	20	0.00000000002293
1	-0.02606629029752	11	0.00000002940887	21	-0.00000000002462
2	0.00173304781808	12	-0.00000001896929	22	0.00000000000666
3	-0.00001952130725	13	0.00000000351784	23	0.00000000000322
4	-0.00006307006450	14	0.00000000204270	24	-0.00000000000354
5	0.00002567895998	15	-0.00000000171995	25	0.00000000000096
6	-0.00000559010027	16	0.00000000039882	26	0.00000000000051
7	-0.00000007279712	17	0.00000000019328	27	-0.00000000000055
8	0.000000065148872	18	-0.00000000019113	28	0.00000000000014
9	-0.00000027698138	19	0.00000000004947	29	0.00000000000009

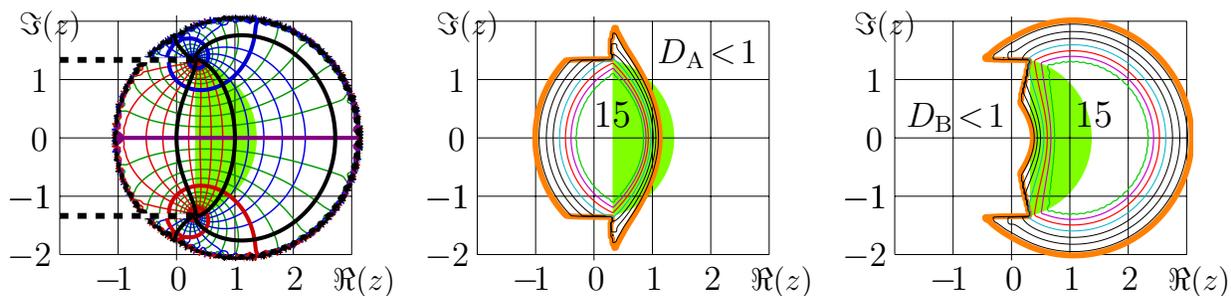


Рис. 10: Approximation $fsl(z)$ by (42), left, and the agreements by (44) and (45); the domain G by (7) is shaded.

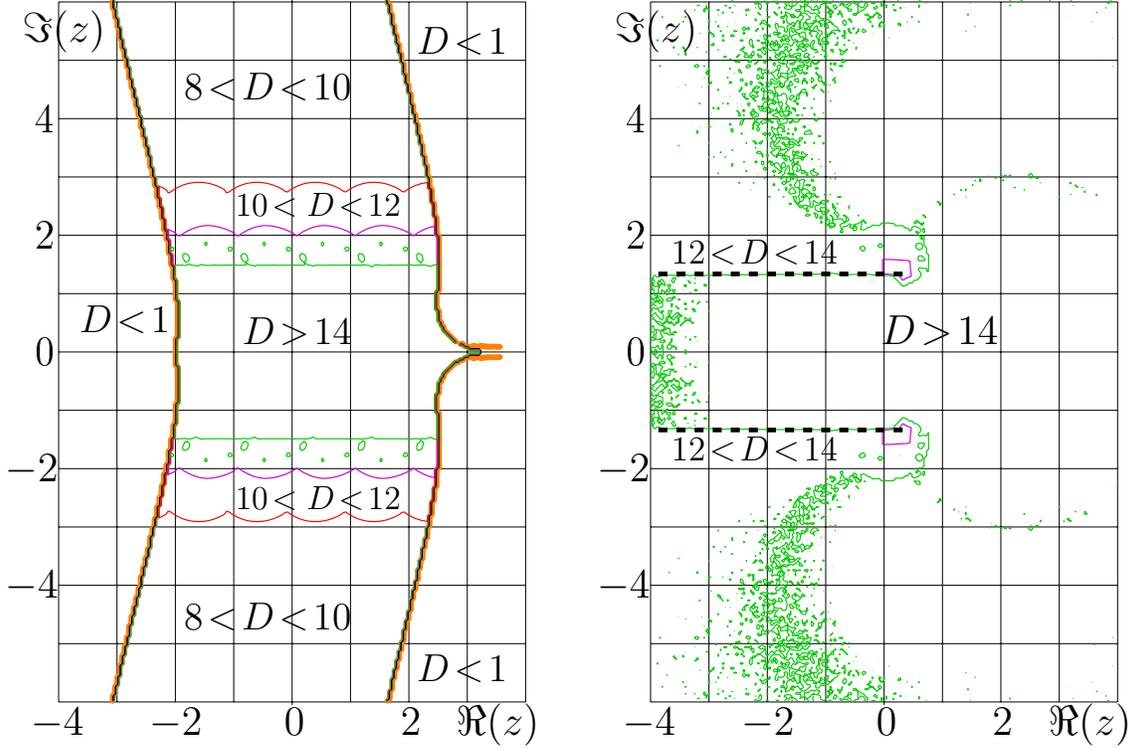


Рис. 11: Agreements of approximations FSE and FSL by (47) and (48).

are plotted in the central and right hand side pictures of figure 10. As before the symbol “15” indicates the region, where the agreement is better than 14. (Within the inner loops, the residuals are smaller that 10^{-14} .)

The range of approximation can be extended with function

$$\text{FSL}(z) = \begin{cases} \text{fsl}(z) & , \quad |\Im(z)| < \Im(L) \ \& \ |z-1| \leq |\log(z)-1| \ \& \ |z-1| \leq |\exp(z)-1| \\ \text{FSL}(\exp(z))-1, & |\Im(z)| < \Im(L) \ \& \ |z-1| > |\exp(z)-1| \\ \text{FSL}(\log(z))+1, & |\Im(z)| \geq \Im(L) \ \text{or} \ |z-1| > |\log(z)-1| \end{cases} \quad (46)$$

Through the extension (46), that the function fsl allows to cover the whole complex plane with the single approximation with the elementary function by (42). In order to check the mutual consistency or approximations FSE and FSL, consider the agreements

$$D_c = -\lg \left| \text{FSL}(\text{FSE}(z)) - z \right| \quad (47)$$

$$D_d = -\lg \left| \text{FSE}(\text{FSL}(z)) - z \right| \quad (48)$$

These agreements are shown in figure 11. The figure confirms the good precision of the approximations. The error of these approximations is comparable to the rounding errors at the complex $\langle \text{double} \rangle$ variables.

7 Conclusion

The numerical algorithm FSE by equaiton (35) approximates the holomorphic tetration (super-exponential) on base e. The algorithm FSL by (46) approximates the inverse function. Up to

date, these are fastest and most precise algorithms. These algorithms can be prototypes for the numerical implementations of tetration and arctetration in compilers of next generation.

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Список литературы

- [1] H. Kneser. “Reelle analytische Lösungen der Gleichung $\varphi(\varphi(x)) = e^x$ und verwandter Funktionalgleichungen”. Journal für die reine und angewandte Mathematik, 1950, **187**, 56-67.
- [2] G. Szekeres, Regular iteration of real and complex functions. Acta Math. 1958, **100**, 203-258.
- [3] G. Szekeres. Fractional iteration of exponentially growing function. J. Austral. Math. Soc., 1961, **2** 301-320.
- [4] N. Bromer. Superexponentiation. Mathematics Magazine, **60** No. 3 (1987), 169-174
- [5] P. Walker. Infinitely differentiable generalized logarithmic and exponential functions. Mathematics of computation, 1991, **196**, 723-733.
- [6] P. Walker. On the solutions of an Abelian functional equation. Journal of mathematical analysis and applications, 1991, **155** No. 1, 93-110
- [7] D. Kouznetsov. Solution of $F(z+1) = \exp(F(z))$ in complex z -plane. Mathematics of Computation, **78** (2009), 1647-1670.
- [8] D. Kouznetsov, H. Trappmann. Portrait of the four regular super-exponentials to base $\sqrt{2}$. Mathematics of Computation, in press. Preprint ILS UEC, 2009, <http://www.ils.uec.ac.jp/~dima/PAPERS/2009sqrt2.pdf>
- [9] H.Trappmann, D.Kouznetsov. Uniqueness of holomorphic Abel functions. Preprint ILS UEC, 2009, <http://www.ils.uec.ac.jp/~dima/PAPERS/2009uniabel.pdf>
- [10] D. Kouznetsov, H. Trappmann. Superfunctions and square root of factorial. Moscow University Physics Bulletin, 2010, in press. Preprint ILS UEC, 2009: <http://www.ils.uec.ac.jp/~dima/PAPERS/2009supefae.pdf>
- [11] Code for fig. 4: <http://en.citizendium.org/wiki/TetrationPolynomial25power.jpg/code>
- [12] Code for fig. 5: <http://en.citizendium.org/wiki/TetrationApproLP100.jpg/code>
- [13] Code for fig. 6: <http://en.citizendium.org/wiki/TetrationTailorExpansion3ipower25.jpg/code>
- [14] Code for fig. 9: <http://en.citizendium.org/wiki/TetrationDerivativesReal.jpg/code>
- [15] Code for fig. 10: <http://en.citizendium.org/wiki/SLOGappro50.jpg/code>