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Superfimctions and sqrt of Factorial

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Abstract—The holomorphic function h is constructed such that $h\,h\,z = z!$; this function is interpreted as square root of Factorial.

Key words: sqrt of Factorial, superfunction, SuperFactorial, inverse problem.
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INTRODUCTION

This work was motivated by one exercise from the past-century course of Quantum Mechanics of the Moscow State University. It was suggested to give sense to the operator $\sqrt{!}$ [1]. That time, a satisfactory solution was not found; the opinion was, that such an operation has no meaning [2].

In Quantum Mechanics, the repeated (iterated) application of an operation (usually, some "observable") to some argument (which may have sense of vector of state) is interpreted as "power" of the operation; in particular, in such a way the square of coordinate or the square of momentum are treated. For this analogy, the notation without parenthesis is used. In these notations, $\sin\alpha$ means $\sin(\alpha)$, $\ln \sin z$ means $\ln(\sin(x))$ and so on; such notations are used also in textbooks on elementary algebra. To avoid confusions at the iterations, we use also the prefix notation Factorial z = Factorial(z) instead of $z!$.

We assume, that factorial is known meromorphic function, just Gamma function [3] with displaced argument. In this way factorial is interpreted in programming languages Mathematica and Maple. The factorial of the real argument is shown in Fig. 1.

In this work, the square root of the Factorial is interpreted as a holomorphic function h such that its second iteration is Factorial, i.e. $hhz = z!$. For real values of argument the graphic of function $\sqrt{!}$ is shown in Fig. 2. Below we describe, how to evaluate it not only for real, but also for complex values of the argument, using superfunctions [4–6].

[1] The English-language version of the article was prepared by the authors.

1. Superfunctions

The evaluation of fractional power of a function, i.e., the fractional iteration, (for example, $\sqrt{\exp}$, see [7–9], or $\sqrt{!}$), can be based on the concept of superfunction [4–6]. For a given function H, which is referred to as the "Transfer Function" below, a superfunction F is a holomorphic solution of the Abel equation

$$F(z+1) = H(F(z)). \quad (1)$$

Such an equation is pretty old [10–12], although in 1827, N.H. Abel wrote it in a different form, for the inverse function of F. The Abel equation comes from the phenomenological consideration of the transfor-

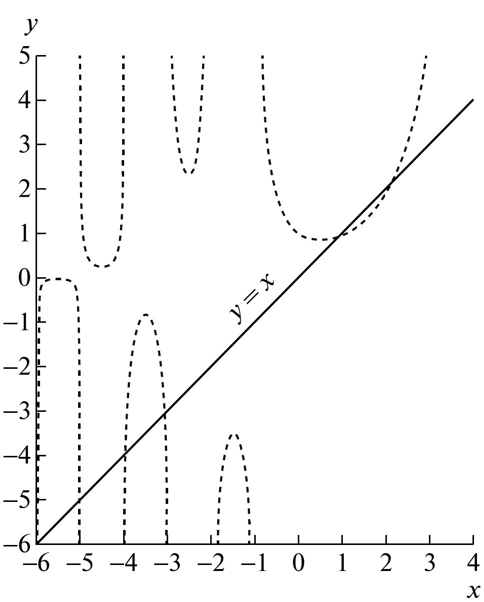

Fig. 1. Factorial of the real argument and graphical solution of equation $x! = x$.

mation of a signal F in a single-dimensional homogenous nonlinear system, characterized with the Transfer Function H. Equation (1) may have also other applications, discussed below in the special section. In some sense, the Eq. (1) is equivalent of the Schröder's equation [13–17]; at the exponential transformation of the argument, the inverse Schröder's function becomes a superfunction, but not every supefunction can be simply expressed through some inverse Schröder function. For this reason, here we deal with superfunctions and not with the Schröder functions.

A superfunction F determines the fractional iteration H^c of the Transfer Function H:

$$H^c(z) = F(c + F^{-1}(z)). \qquad (2)$$

The resulting function H^c can be considered as fractional power of function H, because it satisfies the expected relations

$$H^1 = H, \quad H^{c+d}(z) = H^c H^d z = H^c(H^d(z)),$$

i.e., for two numbers c and d, the identity $H^c H^d = H^{c+d}$ holds, as if H would be not a function but a number. In particular with $c = 1/2$, the half-iteration $h(z) = \sqrt{H}(z) = H^{1/2}(z)$ is considered to be the square root of function H, because $hhz = h(h(z)) = Hz = H(z)$. In this sense $h^2 = H$ and $h = \sqrt{H}$.

Some superfunctions (see Table 1) are well known; they are used without to identify them as superfunctions. Several elementary superfunctions (in particular, trigonometric) were listed also at websites [5, 6, 18].

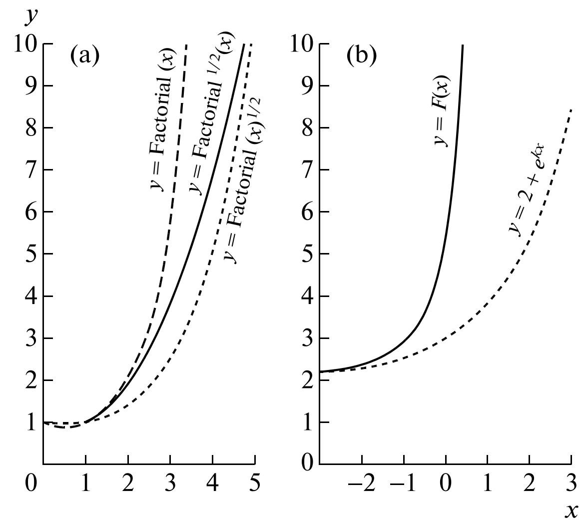

Fig. 2. Left: $y = \text{Factorial}(x)$, thick dashed curve; $y = \sqrt{!}\,(x) = \text{Factorial}^{1/2}(x)$, solid curve; $y = \sqrt{x!} = \text{Factorial}(x)^{1/2}$, thin dashed, as functions of real x; Right: $y = F(x)$, solid, compared to $y = 2+\exp(kx)$, dashed.

Superfunctions of the exponential (see row 4 of the Table 1) yet are not so widely known, although Helmuth Kneser had reported the half-iteration of the exponential, i.e., $\sqrt{\exp}$, in year 1950 [7]. Here, tetrational tet is the superfunction of exp, caracterized in that $\text{tet}_b(0) = 1$ and holomorphie on the range $\mathbb{C}\backslash\{x \in \mathbb{R} : x \le 0\}$; i.e., holomorphie solution of equation

Table 1. Examples of superfunctions

No.	$H(z)$	$F(z)$	$F^{-1}(z)$	Comment
1	$z + 1$	$b + z$	$z - b$	$b \in \mathbb{C}$
2	$b + z$	$bz + c$	$(z - c)/b$	$b \ne 0$
3	$bz + c$	$b^z + \dfrac{c}{1-b}$	$\log_b\left(z - \dfrac{c}{1-b}\right)$	
4	b^z	$\text{tet}_b(z)$	$\text{tet}_b^{-1}(z)$	(3), (4), [4, 9, 19–22]
5	z^b	$\exp(b^z)$	$\log_b(\ln(z))$	$b > 0$
6	$\ln(b + e^z)$	$\ln(bz)$	e^z/b	$b \ne 0$
7	$(a^b + z^b)^{1/b}$	$az^{1/b}$	$(z/a)^b$	$a > 0, b \ne 0$
8	$2z^2 - 1$	$\cos(2^z)$	$\log_2(\arccos(z))$	
9	$2z^2 - 1$	$\cosh(2^z)$	$\log_2(\text{arccosh}(z))$	compare no. 8
10	$2z/(1 - z^2)$	$\tan(2^z)$	$\log_2(\arctan(z))$	
11	$2z/(1 + z^2)$	$\tanh(2^z)$	$\log_2\left(2\ln\left(\dfrac{z+1}{z-1}\right)\right)$	
12	Factorial(z)	SuperFactorial(z)	ArcSuperFactorial(z)	(6), (8)
	$P(H(Q(z)))$	$P(F(z))$	$F^{-1}(Q(z))$	$P(Q(z)) = z$

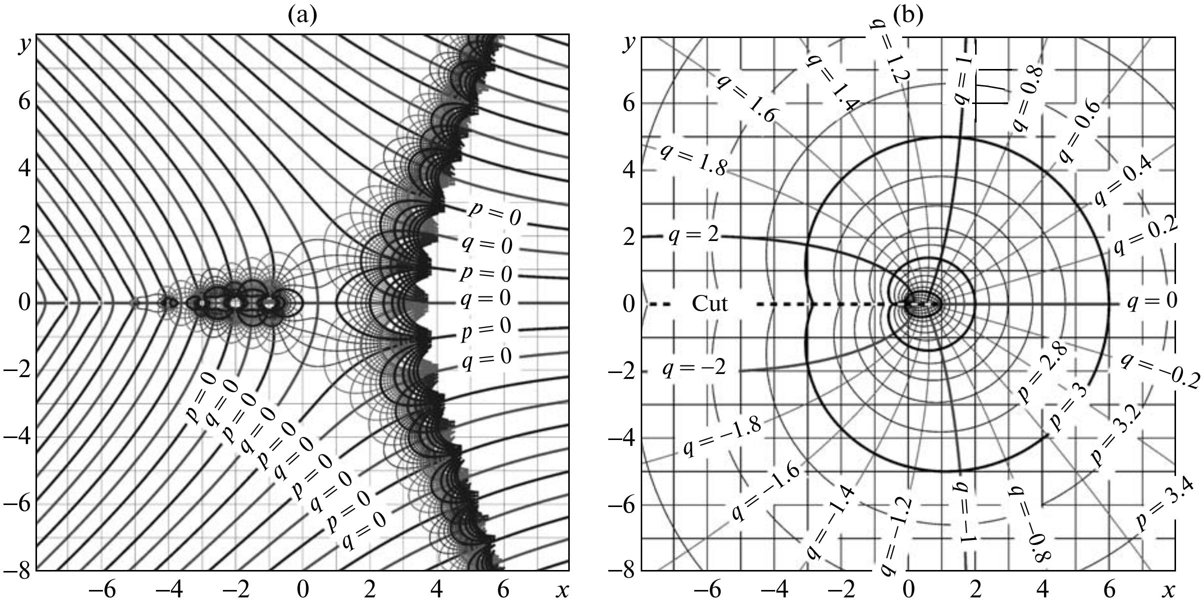

Fig. 3. f = Factorial(z), left, and f = ArcFactorial(z), right, in the complex z-plane; Levels $\Re(f) = p$ = constant and Levels $\Im(f) = q$ = constant are shown with thick lines for $p = -4, -3, -2, -1, 0, 1, 2, 3, 4$ and for $q = -4, -3, -2, -1, 0, 1, 2, 3, 4$.

$$\mathrm{tet}_b(z+1) = \exp_b(\mathrm{tet}_b(z)). \tag{3}$$

At integer values of z, tetration $\mathrm{tet}_b(z)$ is result iterational application of exponential to unity:

$$\mathrm{tet}_b z = \underbrace{\exp_b(\exp_b(..\exp_b}_{z\text{-repetitions of exponential}}(1)..)) \tag{4}$$

Evaluation of tet_b at $b > \exp(1/e)$ and in particular, for $b = e$ and $b = 2$ is described in [9, 19, 22]. For $b = e$, the fast approximation is available at [20, 21]. The evaluation at $1 < b < \exp(1/e)$ and, in particular, for $b = \sqrt{2}$, is considered in [4].

New superfunctions can be obtained by transformation of the already established superfunctions. If some F is SuperFunction of some Transfer Function H, then another superfunctions $\mathcal{F}$ can be define as $\mathcal{F}(z) = F(z + \delta(z))$, where δ is some holomorphic 1-periodic function. In particular, the superfunction in row 9 can be obtained from that in row 8 with $\delta(z) = \pi i \ln(2)/2$ = constant. For more complicated function δ, the resulting function $\mathcal{F}$ usually has either a reduced range of holomorphizm, or some fast growth in the direction of the imaginary axis.

In addition, for the pair of the mutually-inverse functions P and Q, the new pair (Transfer Function, superfunction) can be obtained by the transform, indicated in the last row of the Table 1, from the previous rows. Therefore, the Table 1 can be much larger.

Table 1 can be extended also considering any pair F and F^{-1} of biholomorphic functions, declaring F as superfunction, and constructing the corresponding Transfer Function with (2) at $c = 1$. In this work the inverse problem is considered, i.e., the building of the superfunction for the given Transfer Function, namely, Factorial. This allows to evaluate the holomorphic function $\sqrt{!}$, giving sense to the logo of the Physics department of the Moscow State University.

2. Factorial, ArcFactorial and the Fixed Points

For the evaluation of superFactorial and its inverse, efficient implementations of Factorial and ArcFactorial are required. Complex maps of Factorial and ArcFactorial are shown in Fig. 3. In the right hand side of the left picture, the density of levels for the Factorial is so high that they would overlap; only levels $p = 0$ and $q = 0$ are shown there. We use the original numerical C++ implementation [23] of the Factorial and ArcFactorial.

At the analysis of a superfunction for same Transfer Function H, the key question is about the fixed points [16, 17]. In the case H = Factorial, the fixed points are solutions of Factorial(z) = z. The real fixed points correspond to abscissas of the intersections of the graphics y = Factorial x and $y = x$. These graphics are shown in Fig. 1.

The Factorial(z) has two obvious fixed points $z = 1$ and $z = 2$. However, there is also a countable set of negative fixed points. Two of them are seen in the left bottom corner of Fig. 1.

Each of the fixed points can be used to build-up the superfunction and corresponding non-integer iterations of the transfer function [4]; there exist many functions that can be considered as a square root of the Factorial. In this work, the superfunction built up at the fixed point $z = 2$, and the corresponding $\sqrt{!}$ is considered. Such a choice is determined by the intent to build-up a function that is holomorphic while the argument is larger than 2, and grows up faster than the factorial, which corresponds to the intuitive expectation about such a function. Namely such a factorial could be used for description of processes that grow up faster than any polynomial but slower than any exponential. Namely this $\sqrt{!}$ is plotted in the left hand side of Fig. 2 with a solid curve.

3. Evaluation of SuperFaetorial

Consider the SuperFactorial = F, that approaches the fixed point 2 at large negative values of the real part of the argument:

$$\text{Factorial}(F(z)) = F(z+1), \quad \lim_{x\to-\infty} F(x+\mathrm{i}y) = 2. \quad (5)$$

Following the algorithm [4], we search the solution F in the form

$$F(z) = \Phi(\exp(kz)), \quad (6)$$

where k is constant; function Φ can be interpreted as the inverse Schröder function. The substitution of transformation (6) into Eq. (5) gives the Schröder equation [13, 14]

$$\Phi(K\varepsilon) = \text{Factorial}(\Phi(\varepsilon)), \quad (7)$$

where $K = \exp(k)$. We search the solution Φ in the form

Table 2. Coefficients u and U in the expansions (8) and (9)

n	u_n	U_n
2	0.7987318351724345	−0.7987318351724345
3	0.5778809754764832	0.6980641135593670
4	0.3939788096629718	−0.6339640557572815
5	0.2575339580323327	0.5884152357911399
6	0.1629019581037053	−0.5538887519936520
7	0.1002824191713524	0.5265479025985924
8	0.0603184725913977	−0.5041914604280215
9	0.0355544582258062	0.4854529800293392
10	0.0205859954874424	−0.4694346809094714

$$\Phi(\varepsilon) = 2 + \varepsilon + \sum_{n=2}^{N} u_n \varepsilon^n + \mathbb{O}(\varepsilon^{N+1}), \quad (8)$$

where u are constant real coefficients, and integer $N > 0$. The substitution of such expansion into Eq. (7) gives the value

$$k = \ln(K) = \ln(3 + 2\,\text{Factorial}'(0))$$
$$= \ln(3 - 2\gamma) \approx 0.6127874523307,$$

where $\gamma = -\Gamma'(1) \approx 0.5772156649$ is the Euler constant [3], and the chain of equations for u. From this chain we find

$$u_2 = \frac{\pi^2 + 6\gamma^2 - 18\gamma + 6}{12(3 - 5\gamma + 2\gamma^2)} \approx 0.798731835,$$

$$u_3 = \frac{-36 - 39\pi^2 - 738\gamma^2 + 324\gamma + 99\pi^2\gamma - 60\pi^2\gamma^2 - \pi^4 + 24\gamma^5 + 594\gamma^3 - 120\zeta(3)\gamma + 48\zeta(3)\gamma^2 + 12\gamma^3\pi^2 + 72\zeta(3) - 204\gamma^4}{144(-18 + 69\gamma - 104\gamma^2 + 77\gamma^3 - 28\gamma^4 + 4\gamma^5)}$$

where $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ is the Riemann zeta-function [3]; $\zeta(3) \approx 1.202056903$. Similar (but longer) expressions can be obtained for other coefficients u; few first coefficients are evaluated in Table 2. In addition, we set $u_0 = 2$ and $u_1 = 1$. The partial sum in (8) at $N = 20$ gives the approximation F_{20} for the superFactorial

$$F_{20}(z) = \sum_{n=0}^{20} u_n \exp(knz),$$

providing of order of 15 correct decimal digits at $\Re(z) \le -2$. For evaluation of $F(z)$ at $\Re(z) > -2$, the recursive formula is used, $F(z) = \text{Factorial}(F(z-1))$. Such numerical representation is used to plot the map of SuperFactorial in the complex plane in Fig. 4, as well as the real-real plot in Fig. 2.

The SuperFactorial F is entire periodic function. The period $T = 2\pi\mathrm{i}/k \approx 10.25344968115602792657726406913 97\,\mathrm{i}$ is pure imaginary, so, the pattern in the plot of this function in Fig. 4 is reproduced at the translations along the imaginary axis. Along the real axis, the SuperFactorial F grows up faster, than any exponential (see Fig. 2), and even faster than tetrational [9].

In vicinity of the real axis, the plot of the SuperFactorial shows the quasi-periodic, "fractal" structure, similar to that of growing superexponentials [4, 22]. Perhaps, other holomorphic functions that grow up faster than any finite iteration of exponential (i.e. $\exp^c$ at fixed c) behave in the similar way.

4. ArcSuperFactorial

The inverse function of the SuperFactorial, i.e., ArcSuperFactorial $G = F^{-1}$, is shown in the right hand picture of Fig. 4. This function can be expressed, inverting expansion (8):

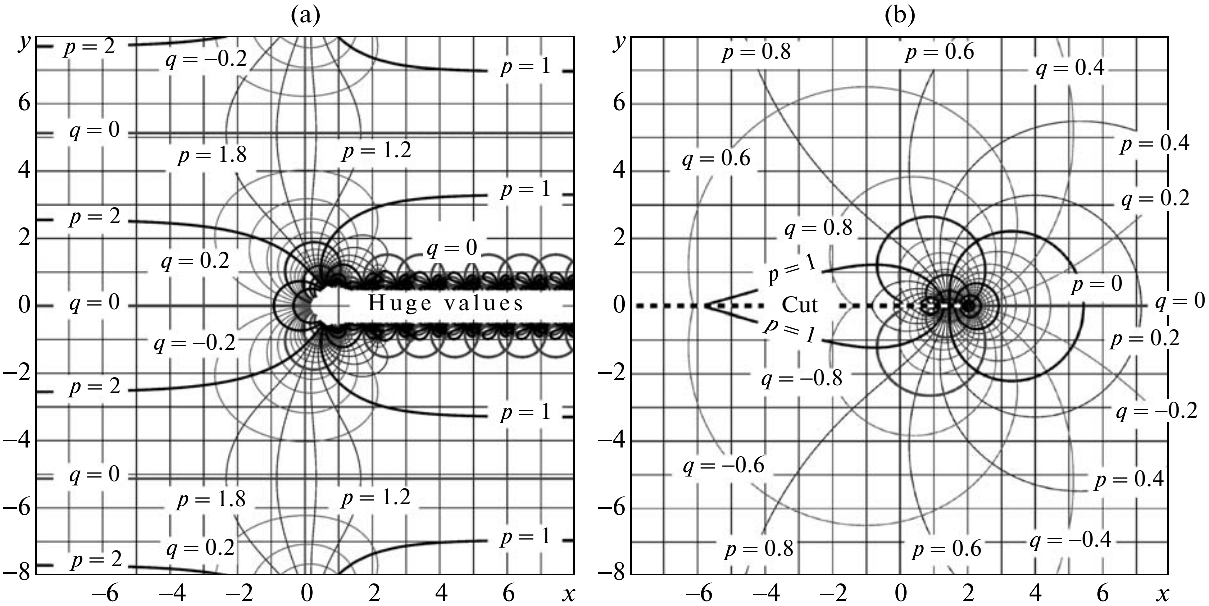

Fig. 4. Maps of $f = F$ and $f = G$ in the same notations as in Fig. 3.

$$G(z) = \frac{1}{k}\log\left(\sum_{n=1}^{N} U_n (z-2)^n + \mathbb{O}(z-2)^{N-1}\right), \qquad (9)$$

where $U_1 = 1$ and $U_2 = -u_2$. The Mathematica routine Inverse Series (see also [3], eq.3.6.25) allows to express coefficients U in terms of coefficients u; both u and U are evaluated in Table 2.

The partial sum in (9) at $N = 20$ is used as approximation for G, allowing to evaluate the $G(z)$ with 15 decimal digits at $|z - 2| \le 0.1$. For other values of the argument, the recurrent equation can be used: $G(z) = G(\text{ArcFactorial}(z)) + 1$. The iterations of ArcFactorial converge to value 2; after several such iterations, the G can be evaluated by expression (9). With complex ⟨d o u b l e ⟩ arithmetics, of order of 14 significant figures can be achieved in such a way.

ArcSuperFactorial $G = F^{-1}$ is holomorphic function on the domain $\mathbb{C}\backslash\{x \in \mathbb{R} : x \le 2\}$. Along the real axis, G grows until infinity, although slowly; slower than any logarithm and even slower than ArcTetrational [9]. Similar slow growth of the modulus of ArcSuperFactorial takes place at the moving out from the origin of coordinates in any direction, except the negative direction of the real axis. Number 2 is branchpoint of G; the cut does left along the real axis.

5. Square Root of Factorial

The choice of the Super Factorial F and ArcSuperFactorial G determines any power of the factorial; in particular, the fractional power and even the complex power. (The Transfer Function can be iterated complex number of times.) If the number of iterations is equal to one half, the combination (2) of SuperFactoral and ArcSuperFactorial gives the $\sqrt{!}$, i.e., the square root of the Factorial.

$$\sqrt{!}\,(z) = \text{Factorial}^{1/2}(z) = F(1/2 + G(z)). \qquad (10)$$

This function in the complex plane is shown in the left hand side of the Fig. 5. For comparison, the right hand picture of Fig. 5 shows the square root of the exponential $\sqrt{\exp}\,(z) = \text{tet}(1/2 + \text{tet}^{-1}(z))$, where tetrational tet is superfunction of the exponential [9], i.e., the holomorphic solution of Eq. (3). The fast numerical implementation of tetrational tet holomorphic in the domain $\mathbb{C}\backslash\{x \in \mathbb{R} : x \le -2\}$, and arctetration tet^{-1} is described in [20]; the Mathematica code for the evaluation tet and tet^{-1} is available at [21]. The Arctetraton tet^{-1} may be called also "superlogarithm" [24], although it is not a superfunction of the logarithm.

The constructed $\sqrt{!}$ is a holomorphic function on the domain $\mathbb{C}\backslash\{x \in \mathbb{R} : x \le 1\}$. In general, $\sqrt{!}$ behaves in a way, similar to $\sqrt{\exp}$; however, it has only one branch point and only one cut line, while the $\sqrt{\exp}$ has two; also, $\sqrt{!}$ grows up a little bit faster, than the $\sqrt{\exp}$. The behavior of $\sqrt{!}$ for real values of the argument is shown in Fig. 2. In general, this function corresponds to the intuitive expectations about its behavior. In particular, it grows faster than any polynomial, but slower than any exponential.

Analogously, maps of function Factorial^c and $\exp^c$ can be plotted for other values c. In particular, at $c = 1$, this function is Factorial; at $c = -1$, it is ArcFactorial,

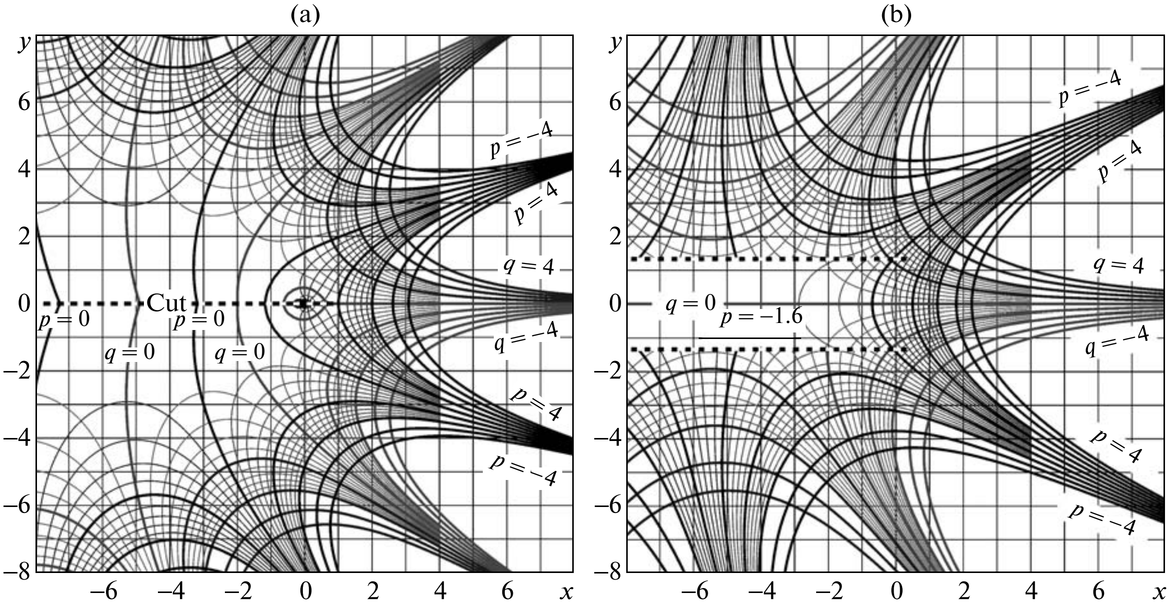

Fig. 5. $f = \sqrt{!}\,(z) = \mathrm{Factorial}^{-1/2}(z)$ by (10) (a), and $f = \sqrt{\exp}\,(z)$ (b), in the complex z-plane in the same notations as in Figs. 3, 4.

shown in Fig. 3; at $c = 0$, H^c becomes identical function.

Equation (2) allows the smooth (holomorphic) transition from the Transfer Function to its inverse function. Function $\sqrt{!}$ appears as the intermediate step at this transition. Such transitions can be defined also for other Transfer Functions.

6. Physical Applications

Although a distributed physical system with the Factorial as Transfer Function is not reported, the same method can be applied for other Transfer Functions. In this section, the possible applications of the formalizm of superfunctions are discussed.

In the investigation of the nonlinear response of optical materials, the sample is supposed to be optically thin, in such a way, that the intensity of the light does not change much as it goes through. Then one can consider, for example, the absorption as function of the intensity. However, at small variation of the intensity in the sample, the precision of measurement of the absorption as function of intensity is not good. The reconstruction of the superfunction from the Transfer Function allows to work with relatively thick samples, improving the precision of measurements. In particular, the Transfer Function of the similar sample, which is half thiner, could be interpreted as the square root (i.e., half-iteration) of the Transfer Function of the initial sample.

In nonlinear acoustics, it may have sense to characterize the nonlinearities in the attenuation of shock waves in a homogeneous tube. This could find an application in some advanced muffler, using nonlinear acoustic effects to withdraw the energy of the sound waves without to disturb the flux of the gas. Again, the analysis of the nonlinear response, i.e., the Transfer Function, may be boosted with the superfunction.

In analysis of condensation, the growth (or vaporation) of a small drop of liqud can be considered, as it diffuses down through a tube with some uniform concentration of wapor. In the first approximation, at fixed concentration of the wapor, the mass of the drop at the output end can be interpreted as the Transfer Function of the input mass. The square root of this Transfer Function will characterize the tube of half length.

In a similar way the mass of a snowball, that rolls down from the hill, can be considered as a function of the path it already have passed. At fixed length of this path (that can be determined by the altitude of the hill) this mass can be considered also as a Transfer Function of the input mass. The mass of the snowball could be measured at the top of the hill and at the bottom, giving the Transfer Function; then, the mass of the snowball as a function of the length it passed is superfunction.

If one neads to build-up an operational element with factorial transfer function, and wants to realize it as a sequencial connection of a couple of identical operational elements, then, each of these two elements should have transfer function $\sqrt{!}$ shown in Fig. 2.

Calculation of a fractal iteration of a function, and in particular, that of exponential or that of the Factorial, may have other (and, perhaps, unexpected) applications; for example, in the description of the physical processes, that grow up faster than any polynomial, but slower than any exponential. The theoretical science should be ready for such applications. In particular, the SuperExponential, SuperFactorial, $\sqrt{\exp}$, $\sqrt{!}$ should be popped up to the level of special functions.

CONCLUSIONS

A SuperFactorial is constructed as superfunction of the Factorial, solving the Eq. (5). An arbitrary power c of a function H can be expressed through its superfunction H by (2). For $H =$ Factorial and $c = 1/2$, this gives way of evaluation of holomorphic function $\sqrt{!}$ shown in Figs. 2 and 5. The formalizm of superfunctions allows the evaluation of non-integer iterations of various functions and may have applications in physics and technology.

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