Recovery of intensity inside a uniform amplifier from its transfer function

Dmitrii Kouznetsov*
IInstitute for Laser Science, University of Electro-Communications,
1-5-1 Chofugaoka, Chofushi, Tokyo, 182-8585, Japan

Propagation of a signal F(x) in a homogeneous unidimensional physical system with longitudinal coordinate x is considered. The piece of length unity of the system is interpreted as a filter, characterized with the transfer function T such that the transfer equation F(x+1)=T(F(x)) holds. The transfer function T is supposed to be known (measured). The problem of reconstruction of function F is considered. Function F is treated as superfunction of function T. The method of regular iterations is suggested as a way to find the solution of the transfer equation. The example with the realistic transfer function T(z)=LambertW(z exp(t+z)) is considered, where t is positive parameter. The solution F(x)=LambertW(exp(t x+1)) has physical meaning, satisfying the equation F'(x)=t F(x)/(1+F(x)) that corresponds to the optical amplifier with saturation, at the initial condition F(0)=1. The hypothesis is formulated that for all distributed ergodic physical systems (amplifiers), the method of regular iterations returns the physically meaningful solution.

CONTENTS

INTRODUCTION

Let some physical system acts as a filter; the signal at the output is determined by the signal at the input. Such a system can be realized as a single - pass optical amplifier, for example, it can be a uniformly pumped active fiber. The system is assumed to be homogeneous, and the signal $F(x)$ inside is expected to be continuous (and even real-holomorphic) function of the longitudinal coordinate x. However, neither back-reflection at the ends of the amplifier, nor backscattering is considered; the structure of the signal in such an approach is ignored.

In the typical case, it is difficult to measure the distribution of the signal inside the system, so, the function F is not known. The dependence of the output signal on the input one, contrary, can be measured, it corresponds to the transfer function T, id est, $T(F(0)) = F(X)$, where X is the length of the amplifier; this length is supposed to be fixed. Without loss of generality, we may set $X = 1$; this means, that the distance from the input to the output is used as unit of length. The amplifiers can be combined; the transfer function of a pair of sequentially coupled amplifiers can be expressed as T^2, that for three amplifiers is T^3 and so on; $T^2(z) = T(T(z))$, $T^3(z) = T(T(T(z)))$, etc..

The problem is, how to reconstruct the distribution of a signal inside the amplifier, if the only the transfer function T is known?

The problem can be formulated as follows: For given T, find a holomorphic solution F of the transfer equation

$$T(F(z)) = F(z+1) \tag{1}$$

The holomorphic solution F is called superfunction for the transfer function T. For $T=\exp$, one of solutions is called tetration, $F=\mathrm{tet}$; $\exp(\mathrm{tet}(z)) = \mathrm{tet}(z+1)$, $\mathrm{tet}(0)=1$. The uniqueness [1, 2] of tetration is provided by the conditions $\mathrm{tet}(z^*) = \mathrm{tet}(z)^*$, $\lim_{y\to+\infty} \mathrm{tet}(x+\mathrm{i}y) = L$, where $L = -\mathrm{LambertW}(-1)^* \approx 0.3+1.3i$ is fixed point of the transfer function, id est, solution of equation $L=\log(L)$. The problem has been formulated in the middle of the past century [3], and in century 21 the unique solution is constructed [1, 2, 4]. Then, the superfunctions were constructed for the exponential on other bases (different from e) [5, 6] and other functions [7, 8].

In this paper, the simple realistic transfer function of the optical amplifier is considered; for this case, both the transfer function and the superfunction can be expressed through the LambertW function. The concept is formulated that the superfunction constructed with the regular iteration [6–8] corresponds to the physically-meaningful solution of equation (1).

REGULAR ITERATION

Assume, T is holomorphic function at least in vicinity of zero; at the careful measurement, several derivatives can be evaluated. Then, the superfunction F can be constructed, following the regular iterations by [6]. This construction is traced below.

Search the solution F of equation (1), that decays to zero at minus infinity, in the following form:

$$F(z) = \varepsilon + a_2\,\varepsilon^2 + a_3\,\varepsilon^3 + ... \tag{2}$$

where $\varepsilon=\mathrm{e}^{kz}$; k and a_2, a_3,.. are constant coefficients; e^k is amplification coefficient of a weak signal. Then

$$F(z+1) = k\varepsilon + a_2\,k^2\,\varepsilon^2 + a_3\,k^3\,\varepsilon^3 + ... \tag{3}$$

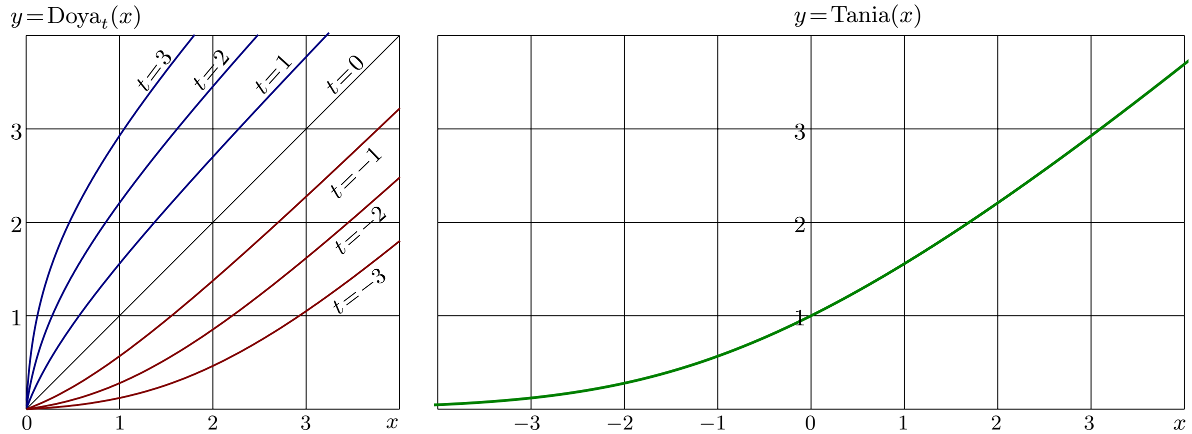

FIG. 1. Explicit plots of functions Doya and Tania by (9).

The substitution of (2) to the left hand side of the transfer equation (1) gives

$$T(F(z)) = T'\varepsilon + T'a_2\varepsilon^2 + T'a_3\varepsilon^3 + .. + \frac{T''}{2}(\varepsilon + a_2\varepsilon^2 + ..)^2 + \frac{T'''}{6}(\varepsilon + ..)^3 + .. \tag{4}$$

where $T' = T'(0)$, $T'' = T''(0)$, $T''' = T'''(0)$; at the careful measurement of the transfer function T and the intelligent fitting, at least the first and the second derivatives can be evaluated. The substitution of (4) and (3) to (1) gives

$$\mathrm{e}^k = T' \tag{5}$$

$$\mathrm{e}^{2k}a_2 = T'a_2 + T''/2 \tag{6}$$

$$\mathrm{e}^{3k}a_3 = T'a_3 + T''a_2 + T'''/6 \tag{7}$$

and so on, determining values of k, a_2, a_3,.. . The representation (2) is good for the small values of ε. At the positive k, this corresponds to the large negative z. Then, for moderate values of z, the precise evaluation is possible with

$$F(z) = T^n\left(F(z-n)\right) \tag{8}$$

for some sufficiently large natural number n. The construction (2)-(8) of superfunction F is called "regular iteration"; it has been suggested for evaluation of the tetration to base $\sqrt{2}$ [6], and then applied to other functions [7, 8].

SIMPLE MODEL

In general, the superfunction (even if the value at some point is specified) is not unique: if F is superfunction, then $\tilde{F}(z) = F(z + \mu(z))$ is also superfunction, while μ is periodic function with period unity. The unuqueness can be provided by the special requirements on the asymptotic behavior of superfunction the complex plane [1, 5, 6, 8], but this refers to the aesthetic reasons rather than to some physical evidence. Therefore, it worth to check that the solution, constructed with the regular iteration, corresponds to some physical quantity at least for some some specific physical system. This section applies the regular iteration to the realistic transfer function, that describes the uniformly-pumped single-pass optical amplifier.

Define two functions, Doya_t and Tania, with

$$\mathrm{Doya}_t(x) = \mathrm{LambertW}\big(x\ \mathrm{e}^{x+t}\big)\ ,\ \ \mathrm{Tania}(x) = \mathrm{LambertW}(\exp(x+1)) \tag{9}$$

where t is real parameter. These functions are described in [9, 10] and plotted in figure 1.

Consider the amplifier with the transfer function

$$T(z) = I_{\mathrm{sat}}\mathrm{Doya}_t(z/I_{\mathrm{sat}}) \tag{10}$$

where I_{sat} is constant, that has sense of the saturation intensity of the uniformly pumped gain medium.

Then, the regular iteration of the previous section gives the expansion of the superfunction,

$$F(x) = I_{\mathrm{sat}} \left(q - q^2 + \frac{3}{2} q^3 + ... \right) \tag{11}$$

where $q = \exp(tx)$. Actually $F(x)$ is distribution of the intensity along the coordinate x of propagation of signal in the uniformly pumped amplifier; the distance is measured in the units of length of the amplifier. The parameter t has no need to be positive; the negative values correspond to the saturable absorber.

For the transfer function by (10), the superfunction can be expressed through the Tania function by (9) as follows:

$$F(x) = I_{\mathrm{sat}} \mathrm{Tania}\big(t(x+1)\big) \tag{12}$$

The Tania function satisfies the simple differential equation

$$\mathrm{Tania}'(x) = \mathrm{Tania}(x) / \big(1 + \mathrm{Tania}(x)\big) \tag{13}$$

which appears also in the simple model of the medium [11] and waves propagating inside [12].

This should be considered as verification of the method of regular iterations for the realistic amplifier, while both, the transfer function and the superfunction can be expressed through the special function LambertW.

The application of the method for recovery of superunctions from the measured transfer functions may be continuation of this work.

CONCLUSION

The method of regular iterations allows to recover the distribution of intensity inside the amplifier from its transfer function T, giving the superfunction F by eqs. (2)(8). For the model transfer function (10), the result agrees with the analytic expression (12); in this case both the transfer function T and the superfunction F can be expressed through the LamertW function. This confirmation allows to formulate the general concept, conjecture:

For any homogeneous distributed amplifier, characterized with the transfer function T, the distribution of intensity inside is determined by the superfunction F constructed by the regular iteration through (2)-(8).

This concept is not trivial, because, in general, the superfunction is not unique. In principle, the concept above could be rejected with the detailed numerical simulations and/or precise measurements of the transfer function T for some amplifier and comparison of the superfunction F by (2),(8) to the direct simulation of measurement of the intensity inside the amplifier. The author expects, that the concept is valid for all homoheneous amplifiers characterized with the transfer function.

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* dima@ils.uec.ac.jp

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