Transfer function of an amplifier and characterization of Materials

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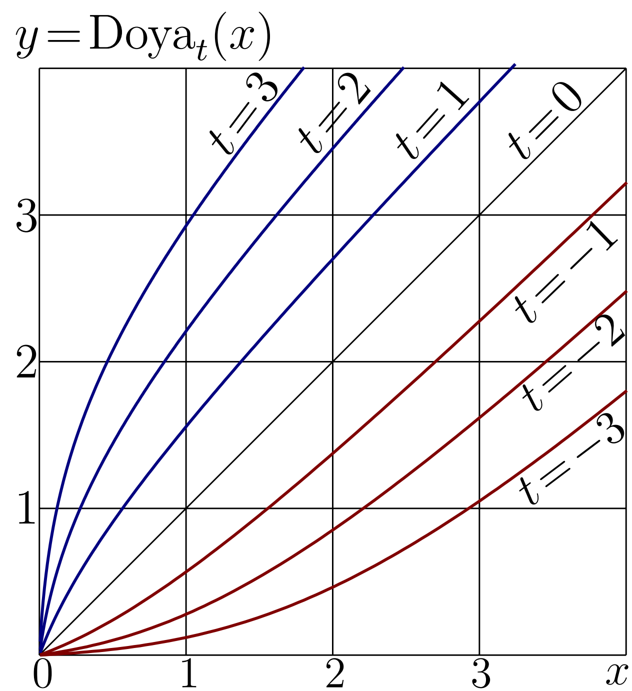
Transfer function

Consider some uniformly pumped amplifier. Assume, the output intensity is determined as some function of the input intensity. Let us call this function **Transfer function**:

$$I_{\text{output}} = T(I_{\text{input}})$$



Example:

$$T(I) = \mathrm{Doya}_t(I/I_{\text{sat}})$$

where

$$\mathrm{Doya}_t(x) = \mathrm{LambertW}(x\ \mathrm{e}^{x+t})$$

t is unsaturated gain per length of amplifier;

for $I < I_{\text{sat}}$, the exponential dependence:

$I_{\text{out}} = I_{\text{in}} \exp(t)$

$$\mathrm{ArcLambertW}(x) = \mathrm{LambertW}^{-1}(x) = x\,\mathrm{e}^x$$

Question: If we know the transfer function T, can we reconstruct $I(z)$?

$I(z+1) = T(I(z))$ is called "transfer equation".

Transfer equation: $F(z\!+\!1) = T(F(z))$

where z is distance along the amplifier measured in units of its length.
Assume, T is known. Is it possible to get F ?

Regular iteration. Let $T(0)\!=\!0$, $\varepsilon = \mathrm{e}^{kz}$

$$F(z) = \varepsilon + a_2\, \varepsilon^2 + a_3\, \varepsilon^3 + ...$$

Then $F(z\!+\!1) = \mathrm{e}^k \varepsilon + a_2\, \mathrm{e}^{2k} \varepsilon^2 + a_3\, \mathrm{e}^{3k} \varepsilon^3 + ...$

$$T(F(z)) = T'\varepsilon + T' a_2 \varepsilon^2 + T' a_3 \varepsilon^3 +..+ \tfrac{T''}{2}(\varepsilon\!+\!a_2\varepsilon^2\!+..)^2 + \tfrac{T'''}{6}(\varepsilon\!+..)^3 +..$$

$$T(F(z)) = T'\varepsilon + \left(T' a_2 + \tfrac{T''}{2}\right)\varepsilon^2 + \left(T' a_3 + T'' a_2 + \tfrac{T'''}{6}\right)\varepsilon^3 + ...$$

$$\mathrm{e}^k = T' \;;\quad \mathrm{e}^{2k} a_2 = T' a_2 + \tfrac{T''}{2} \;;\quad \mathrm{e}^{3k} a_3 = T' a_3 + T'' a_2 + \tfrac{T'''}{6} \;;\quad ...$$

$$k = \ln(T') \;;\quad a_2 = \frac{T''/2}{\mathrm{e}^{2k} - T'} \;;\quad a_3 = \frac{T'' a_2 + T'''/6}{\mathrm{e}^{3k} - T'} \;;\quad ...$$

The expansion is for $z \to -\infty$; the approximation is precise at $\varepsilon \ll 1$;

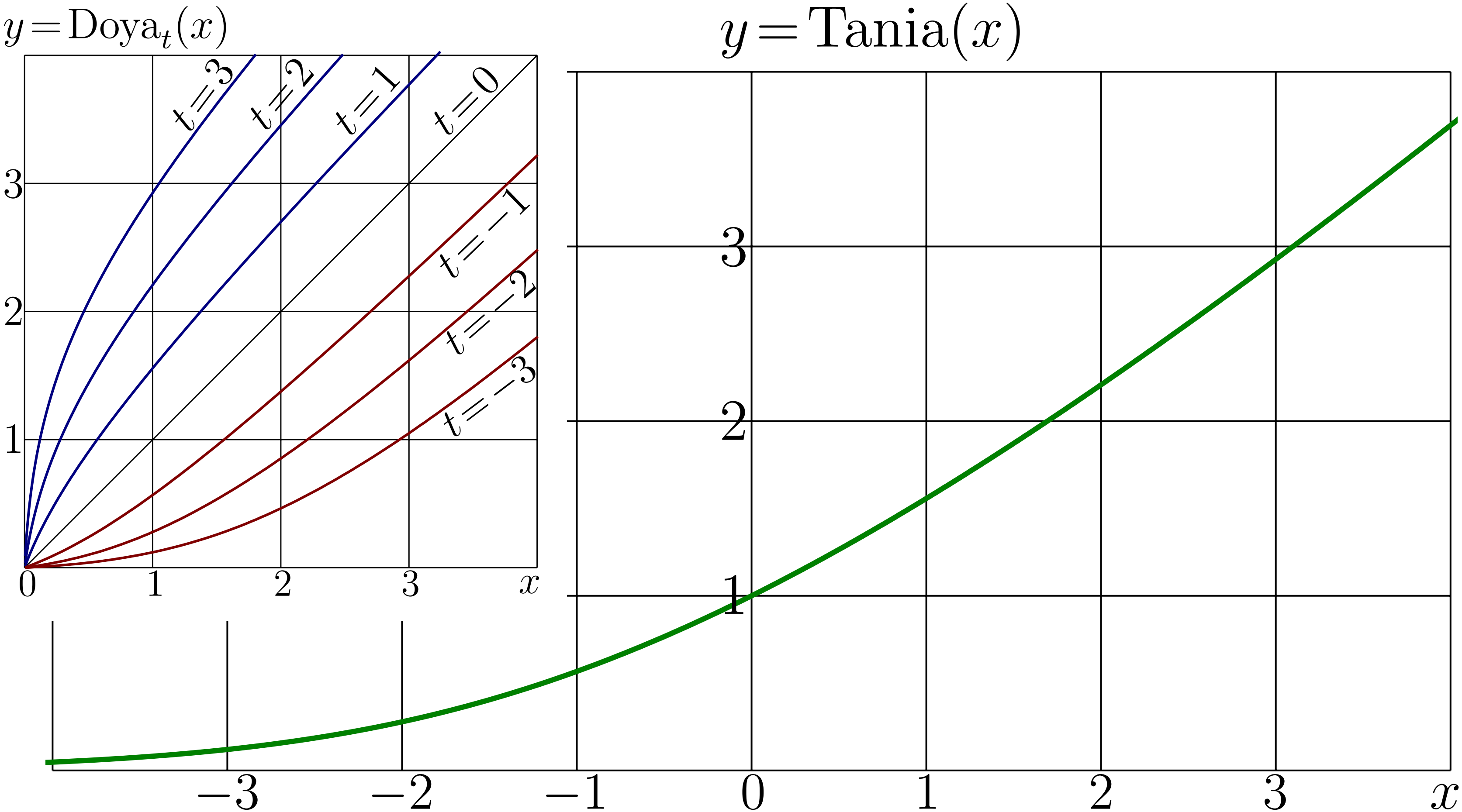
if not, use $F(z) = T(F(z\!-\!1)) = T(T(F(z\!-\!2))) = T^n(F(z\!-\!n))$

The **Regular iteration** gives way of evaluation of the superfunction F for a transfer function T with fixed point 0.

Example of solution of Transfer equation $\quad F(z+1) = T(F(z))$

$$T(I) = \mathrm{Doya}_t(I/I_{\mathrm{sat}}) \quad , \quad F(x) = I_{\mathrm{sat}} \mathrm{Tania}(t\, x)$$

$$\mathrm{Tania}(x) = \mathrm{LambertW}(\exp(x+1))$$



Physical meaning of Tania and Doya functions

Tania function is solution $f = \mathrm{Tania}$ of $\quad f'(z) = \dfrac{f(z)}{1+f(z)}$,

with additional confition $f(0)=1$.
The efficient C++ algorithm of the evaluation is suggested at
`http://tori.ils.uec.ac.jp/TORI/index.php/Tania_function`

Then, $\mathrm{Doya}_t(z) = \mathrm{Tania}(t + \mathrm{ArcTania}(z))$

The Tania function appears as superfunction of Doya_1;
The ArcTania appears as its Abel function, satisfying the Abel equation

$$\mathrm{ArcTania}(\mathrm{Doya}_1(z)) = \mathrm{ArcTania}(z) + 1$$

Explicit form:

$$\mathrm{Doya}_t(x) = \mathrm{LambertW}\Big(x\ \mathrm{e}^{x+t}\Big)$$

$$\mathrm{Tania}(x) = \mathrm{LambertW}(\exp(x+1))$$

$$\mathrm{ArcTania}(z) = z + \ln(z) - 1$$

	$T(z)$	$F(z)$	$G(z)=F^{-1}(z)$
1	$z+1$	$b+z$	$z-b$
2	$b+z$	$bz+c$	$(z-c)/b$
3	$bz+c$	$b^z+\frac{c}{1-b}$	$\log_b\left(z-\frac{c}{1-b}\right)$
4	b^z	$\mathrm{tet}_b(z)$	$\mathrm{tet}_b^{-1}(z)$
5	z^b	$\exp(b^z)$	$\log_b(\ln(z))$
6	$\ln(b+\mathrm{e}^z)$	$\ln(bz)$	e^z/b
7	$(a^b+z^b)^{1/b}$	$az^{1/b}$	$(z/a)^b$
8	$2z^2-1$	$\cos(2^z)$	$\log_2(\arccos(z))$
9	$2z^2-1$	$\cosh(2^z)$	$\log_2(\mathrm{arccosh}(z))$
10	$2z/(1-z^2)$	$\tan(2^z)$	$\log_2(\arctan(z))$
11	$2z/(1+z^2)$	$\tanh(2^z)$	$\log_2\left(2\ln\frac{z+1}{z-1}\right)$
12	$z!$	$\mathrm{SuperFactorial}(z)$	$\mathrm{AbelFactorial}(z)$
13	$\mathrm{Doya}_t(z)$	$\mathrm{Tania}(tz)$	$\mathrm{ArcTania}(z)/t$
	$P(H(P^{-1}(z)))$	$P(F(z))$	$G(P^{-1}(z))$

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Dictionary:

http://tori.ils.uec.ac.jp/Abel_equation.htm

http://tori.ils.uec.ac.jp/Abel_function.htm

http://tori.ils.uec.ac.jp/Doya_function.htm

http://tori.ils.uec.ac.jp/LambertW_function.htm

http://tori.ils.uec.ac.jp/Regular_iteration.htm

http://tori.ils.uec.ac.jp/Superfunction.htm

http://tori.ils.uec.ac.jp/Table_of_superfunctions.htm

http://tori.ils.uec.ac.jp/Tania_function.htm

http://tori.ils.uec.ac.jp/Tetration.htm

http://tori.ils.uec.ac.jp/Transfer_equation.htm

http://tori.ils.uec.ac.jp/Transfer_function.htm

TORI means **T**ools for **O**utstanding **R**esearch and **I**nvestigation
The Colleagues who are interested to use these tools are invited as readers.
The Colleagues who are interested to develop these tools are invited as writers.

CONCLUSIONS

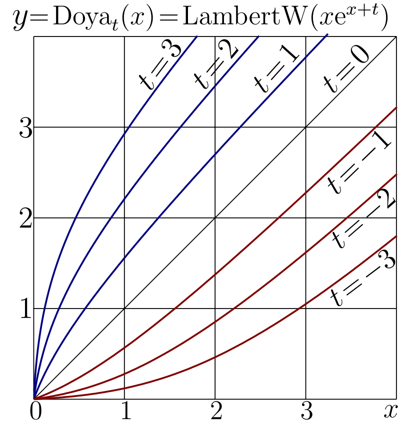
Optical amplifier is characterized with the transfer function T.

$$I_{\text{out}} = T(I_{\text{in}})$$

Example of realistic transfer function: $T(I) = \text{Doya}_t(I/I_{\text{sat}})$

The **Regular iteration** method is suggested for the recovery of $I(z)$ from the transfer function,

`http://tori.ils.uec.ac.jp/Regular_iteration.htm`

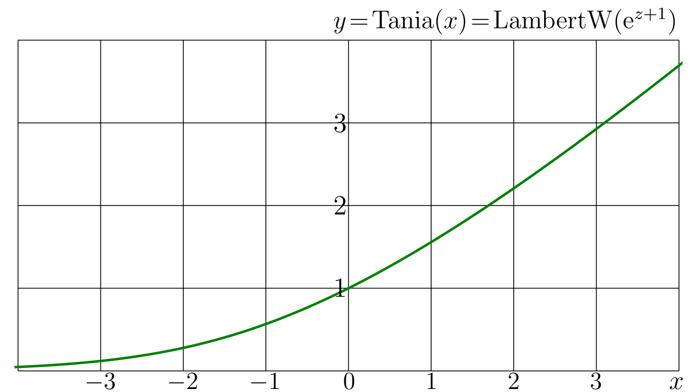


For the model transfer function, $I(z)$ is expressed in terms of special functions:

$$I(z) = I_{\text{sat}} \text{Tania}(tz) = I_{\text{sat}} \, \text{LambertW}(e^{tz+1})$$

In general, I appears as **Superfunction** of T.

`http://tori.ils.uec.ac.jp/Superfunction.htm`



The formalism is expected to be useful in the design of the cascade amplifiers and in the characterization of the laser materials (ceramics, crystals, fibers).

The comparison of reconstructed functions with experimental data should be greatly appreciated.

Acknowledgement

The formalism of superfunctions is developed in collaboration with **Henryk Trappmann**