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COMPUTATION OF THE TWO REGULAR SUPER-EXPONENTIALS TO BASE EXP(1/E)

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ABSTRACT. The two regular super-exponentials to base exp(1/e) are constructed. An efficient algorithm for the evaluation of these super-exponentials and their inverse functions is suggested and compared to the already published results.

1. INTRODUCTION

We call a holomorphic function F a super-function [10] of some base function f if it is a solution of the equation

$$F(z+1) = f(F(z)), \tag{1}$$

In the case $f = \exp_b$, i.e., for the exponential base function $f(z) = b^z$, we call F *super-exponential* to base b. In addition, if the super-exponential F satisfies the equation

$$F(0) = 1, \tag{2}$$

we call it *tetrational*; for integer values of the argument z, equation, (1) and (2) imply F to be the z times application of the exponential $\exp_b$ to unity

$$F(z) = \underbrace{\exp_b\Big(\exp_b(\dots \exp_b}_{z\times}(1)\dots)\Big). \tag{3}$$

Conversely, a function A is called an Abel function of some base function f if it satisfies

$$A(f(z)) = A(z)+1. \tag{4}$$

For $f(z) = b^z$ we call A *super-logarithm* to base b. The inverse of a super-exponential is a super-logarithm. (In some ranges of values of z, the relations $F(A(z)) = z$ and $A(F(z)) = z$ hold.)

We have constructed super-exponentials and efficient algorithms of their numerical evaluation in [7] for $1<b<\mathrm{e}^{1/\mathrm{e}}$, and in [9] for $b>\mathrm{e}^{1/\mathrm{e}}$. However, neither method used in these publications is applicable to base $b = \mathrm{e}^{1/\mathrm{e}}$. This case is especially analyzed by Walker in [18]; he evaluates the two Abel functions at several points in the complex plane. Here we show that his constructions are equal to the two regular Abel functions (*regular* in the sense of Szekeres [15]) and suggest a faster/more precise alternative algorithm (which goes back to Écalle) that allows us to plot the complex maps in real time.

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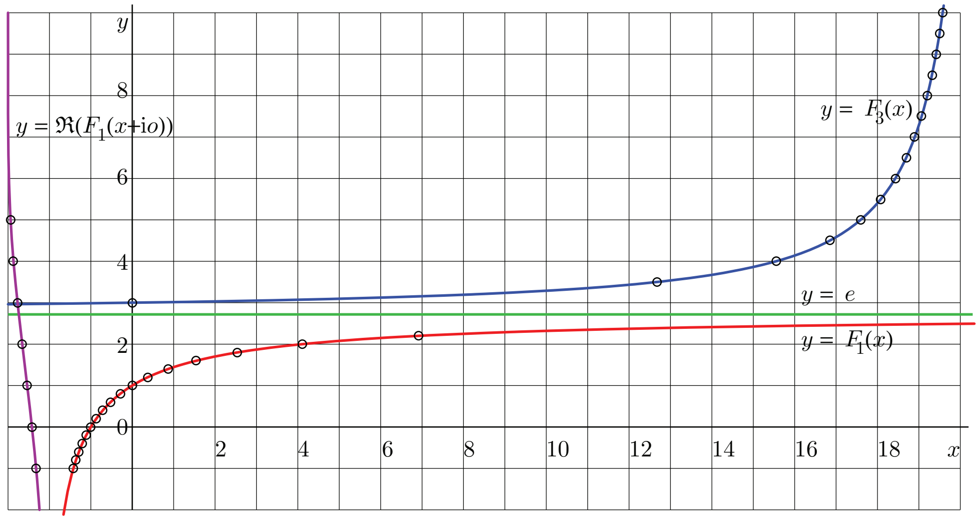


FIGURE 1. Super-exponentials to base $b = \exp(1/\mathrm{e})$ versus real argument; the asymptotic $y = \mathrm{e}$; circles represent the data from [18].

The two super-exponentials F_1 and F_3 along the real axis are shown in Figure 1. The circles represent the data from Tables 1 and 3 by [18]. The behavior of these functions and their inverses in the complex plane is shown in Figure 2.

As in [9], the subscript of the super-exponential (here 1 or 3) indicates the value at 0 of the chosen representative of the class of all super-exponentials obtained by argument shift $F(z+c)$, $c \in \mathbb{C}$. We often identify this whole class as one super-exponential. We consider two classes of super-exponentials represented by F_1 with $F_1(0) = 1$ and by F_3 with $F_3(0) = 3$, respectively. According to the definition, F_1 is a tetrational. For the other (above unbounded) super-exponential, the smallest integer from the range of values along the real axis is chosen as value at zero.

2. FOUR METHODS OF CALCULATING THE REGULAR ITERATION WITH MULTIPLIER 1

In the theory of regular iteration (see e.g. [15] or [12]) there are several algorithms available to compute the regular fractional/continuous iteration and the Abel function of an analytic function at the fixed point 0. Functions h with multiplier 1, e.g., $h'(0) = 1$, are treated differently from functions h with $|h'(0)| \neq 0, 1$.

In our case we have the base function $f(z) = \mathrm{e}^{z/\mathrm{e}}$ with fixed point e and $f'(\mathrm{e}) = 1$. As the whole theory of regular iteration assumes the fixed point to be at 0, we move the fixed point to 0 via a conjugation with the linear transformation τ: Let $\tau(z) = \mathrm{e}\,(z+1)$, then $\tau^{-1}(z) = z/\mathrm{e} - 1$ and

$$\tau^{-1} \circ f \circ \tau(z) = \mathrm{e}^z - 1 =: h(z),$$
$$f = \tau \circ h \circ \tau^{-1}.$$

The regular iterates $f^{[t]}$ at the fixed point e are then given by $f^{[t]} = \tau \circ h^{[t]} \circ \tau^{-1}$, where the regular iterates of h can be obtained in one of the ways described later.

TABLE 1. Computing the Abel function with Lévy's formula.

n	y_n	n	y_n	n	y_n	n	y_n
100	−1.4560	1,000	−1.425788	10,000	−1.4226982	100,000	−1.42241848
101	−1.4557	1,001	−1.425785	10,001	−1.4226982	100,001	−1.42241893
102	−1.4553	1,002	−1.425781	10,002	−1.4226981	100,002	−1.42241823
103	−1.4550	1,003	−1.425778	10,003	−1.4226981	100,003	−1.42241951
104	−1.4547	1,004	−1.425775	10,004	−1.4226981	100,004	−1.42241880
105	−1.4544	1,005	−1.425771	10,005	−1.4226980	100,005	−1.42241891
106	−1.4541	1,006	−1.425768	10,006	−1.4226980	100,006	−1.42241937
107	−1.4538	1,007	−1.425764	10,007	−1.4226980	100,007	−1.42241983
108	−1.4535	1,008	−1.425761	10,008	−1.4226979	100,008	−1.42241913
109	−1.4533	1,009	−1.425758	10,009	−1.4226979	100,009	−1.42241958.

The regular Abel function α of h (up to an additive constant determined by u) is defined by the inverse of $\sigma_u(t) = h^{[t]}(u)$. We call this σ_u the regular super-function of h with $\sigma_u(0) = u$:

$$\sigma_u(t) := h^{[t]}(u) \qquad \alpha_u := \sigma_u^{-1} \qquad h^{[t]}(z) = \sigma_u(t + \alpha_u(z)). \tag{5}$$

The regular Abel function A_u of f at e with $A_u(u) = 0$ and the regular super-function F_u of f at e with $F_u(0) = u$ can be obtained by

$$A_u = \alpha_{\tau^{-1}(u)} \circ \tau^{-1} \qquad F_u = \tau \circ \sigma_{\tau^{-1}(u)}. \tag{6}$$

The classic limit formula of Lévy [13] (see also Kuczma [12], Theorem 3.5.6) for the regular Abel functions of h with multiplier 1 is:

$$\alpha_u(z) = \lim_{n\to\infty} \frac{h^{[n]}(z) - h^{[n]}(u)}{h^{[n+1]}(u) - h^{[n]}(u)}, \tag{7}$$

$$A_u(z) = \lim_{n\to\infty} \frac{f^{[n]}(z) - f^{[n]}(u)}{f^{[n+1]}(u) - f^{[n]}(u)}. \tag{8}$$

One can verify that in our case of $h(z) = e^z - 1$, Lévy's formula converges just too slowly; it is difficult to reach sufficient precision to make any camera-ready plot of the Abel function. In Table 1 we display

$$y_n = \frac{f^{[n]}(-1) - f^{[n]}(1)}{f^{[n+1]}(1) - f^{[n]}(1)} \to A_1(-1).$$

There is another interesting possibility to compute the regular super-function, which we call here *Newton limit formula* (probably first mentioned by Écalle in [2]) because of its similarity to the Newton binomial series of $x^t = (1 + (x-1))^t$:

$$\sigma_u(t) = h^{[t]}(u) = \sum_{n=0}^{\infty} \binom{t}{n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} h^{[m]}(u), \tag{9}$$

$$F_u(t) = f^{[t]}(u) = \sum_{n=0}^{\infty} \binom{t}{n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} f^{[m]}(u). \tag{10}$$

However, this method also has a depressing slow convergence; moreover, we need a bigger internal precision caused by the involved summation. For example, for 1000 summands and 2000 bits precision with $u = 1$ and $t = -1.4223536677333$ we get $F_u(t) \approx -0.9875$ while we would expect a value very close to -1 (see Table 1).

TABLE 2. Computing the Abel function with Fatou's formula.

n	a_n	n	a_n	n	a_n
1,000	−1.4224939	10,000	−1.422367740	100,000	−1.42235507550
1,001	−1.4224938	10,001	−1.422367738	100,001	−1.42235507549
1,002	−1.4224936	10,002	−1.422367737	100,002	−1.42235507548
1,003	−1.4224935	10,003	−1.422367736	100,003	−1.42235507546
1,004	−1.4224934	10,004	−1.422367734	100,004	−1.42235507545
1,005	−1.4224932	10,005	−1.422367733	100,005	−1.42235507543

Another formula to compute an Abel function of $\mathrm{e}^x - 1$ is given in Walker's text [18]. He computes an Abel function g_1 of $a(z) = 1 - \mathrm{e}^{-z}$ and an Abel function g_2 of h with a formula which goes back to Fatou [4]:

$$g_1(z) = \lim_{n\to\infty} -\frac{1}{3}\log(n) + \frac{2}{a^{[n]}(z)} - n, \quad z < 0. \tag{11}$$

We derive the corresponding Abel function of h by knowing that $a(z) = -h(-z)$.

$$\alpha_{\mathrm{W}}^{(1)}(z) = g_1(-z) = \lim_{n\to\infty} -\frac{1}{3}\log(n) - \frac{2}{h^{[n]}(z)} - n, \quad z < 0, \tag{12}$$

$$\alpha_{\mathrm{W}}^{(2)}(z) = g_2(z) = \lim_{n\to\infty} -\frac{1}{3}\log(n) - \frac{2}{h^{[-n]}(z)} + n, \quad z \geq 0. \tag{13}$$

The convergence of this formula is better than that of Lévy but still rather slow (which Walker notices too and that's why he introduces a slightly accelerated version, which we omit here for brevity). To have an impression of the convergence of Fatou's/Walker's formula, we display

$$\begin{aligned} y_n &:= -\frac{2}{h^{[n]}(-\mathrm{e}^{-1}-1)} + \frac{2}{h^{[n]}(-1)} - 1 \\ &\longrightarrow \alpha_{\mathrm{W}}^{(1)}(\tau^{-1}(-1)) - \alpha_{\mathrm{W}}^{(1)}(\tau^{-1}(0)) - 1 = A_1(-1) \end{aligned}$$

in Table 2.

Before we give the fourth method and show that Walker's formula is equivalent to it, we start with some formal background about regular iteration.

Definition 1 (regular iteration)**.** For every formal powerseries

$$h(z) = z + \sum_{n=m}^{\infty} h_n z^n, \quad m \geq 2, h_m \neq 0 \tag{14}$$

and each $t \in \mathbb{C}$ there is exactly one formal powerseries $h^{[t]}(z) = z + \sum_{n=m}^{\infty} {h^{[t]}}_n z^n$, such that ${h^{[t]}}_m = t \cdot h_m$ and $h^{[t]} \circ h = h \circ h^{[t]}$. We call $h^{[t]}$ the regular iteration of h. It satisfies $h^{[1]} = h$ and $h^{[s+t]} = h^{[s]} \circ h^{[t]}$ and is given by the formula:

$${h^{[t]}}_N = \sum_{n=0}^{N-1} \binom{t}{n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} {h^{[m]}}_N \tag{15}$$

$$= \sum_{m=0}^{N-1} (-1)^{N-1-m} \binom{t}{m} \binom{t-1-m}{N-1-i} {h^{[m]}}_N \tag{16}$$

where (16) can already be found as formula (2.19) in [5].

The formal powerseries $h^{[t]}$ is not necessarily convergent even if h is. We call a function which has the powerseries $h^{[t]}$ an asymptotic expansion at 0 a regular iteration of h.

If $z \mapsto h^{[t]}(z)$ is an analytic function in some domain, we call the function $\sigma(t) = h^{[t]}(z_0)$ a regular super-function of h for any z_0 in the domain, and we call its inverse a regular Abel function of h. Usually we identify Abel functions that only differ by a constant and we identify superfunctions that are translations of each other (i.e. $x \mapsto \sigma(x+c)$ is identified with σ).

Definition 2. Let h be a formal powerseries of the form $h(x) = x + \sum_{k=m}^{\infty} h_k x^k$, $h_m \neq 0$. Its iterative logarithm is the unique formal powerseries j of the form $j(x) = \sum_{k=m}^{\infty} j_k x^k$ with $j_m = h_m$ that satisfies the Julia equation

$$j \circ h = h' \cdot j. \tag{17}$$

One obtains the Julia equation when differentiating the Abel equation and then substituting $\alpha' = 1/j$. For reference we give the first few coefficients of the iterative logarithm j of $h(x) = \mathrm{e}^x - 1$:

$$j(x) = \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{48}x^4 - \frac{1}{180}x^5 + \frac{11}{8640}x^6 - \frac{1}{6720}x^7 + \dots.$$

From this iterative logarithm one can get a description of the regular Abel function by $\alpha = \int \frac{1}{j}$.

$$\alpha'(x) = \frac{1}{j(x)} = 2x^{-2} + \frac{1}{3}x^{-1} - \frac{1}{36} + \frac{1}{270}x + \frac{1}{2592}x^2 - \frac{71}{108864}x^3 + \dots.$$

If we integrate this to get α, the term x^{-1} becomes $\log|x|$ for real x and $\log(\pm x)$ for complex values of x; the choice of the sign determines the branch of the resulting function (that unavoidably has the cutline). This gives the expansion

$$\alpha(x) = \underbrace{-2x^{-1} + \frac{1}{3}\log(\pm x)}_{s(x)} \underbrace{- \frac{1}{36}x + \frac{1}{540}x^2 + \frac{1}{7776}x^3 - \frac{71}{435456}x^4 + \dots}_{v(x)}. \tag{18}$$

This formula cannot be used to get *arbitrary* precision, because $v(x)$ is not convergent as we show now; however, suitable truncation of the divergent series can be used to obtain a *certain* precision.

Preliminary 1 (Baker 1958 [1] Satz 17)**.** *The regular iteration $h^{[t]}$ of $h(x) = \mathrm{e}^x - 1$ has non-zero convergence radius exactly if t is an integer.*

Preliminary 2 (Écalle 1975 [3])**.** *Let h be a formal powerseries with multiplier 1. Its regular iteration powerseries $h^{[t]}$ has a positive radius of convergence for all $t \in \mathbb{C}$ if and only if its iterative logarithm has a positive radius of convergence.*

Theorem 3. *The formal powerseries $v(z)$ in* (18) *has 0 convergence radius.*

Proof. Suppose that $v(z)$ has non-zero radius of convergence. Then also $v'(z) = \alpha'(z) - s'(z)$ has non-zero radius of convergence. Then $z^2 v(z)$ has non-zero radius of convergence and then $z^2 s'(z) + z^2 v(z)$ is a powerseries with non-zero radius of convergence with non-zero zeroth coefficient. Then

$$j(z) = \frac{1}{\alpha'(z)} = \frac{z^2}{z^2 s'(z) + z^2 v(z)}$$

has non-zero radius of convergence. Then by Theorem 2 the regular iteration $h^{[t]}$ of $h(x) = \mathrm{e}^x - 1$ has non-zero radius of convergence for all t. This is in contradiction to theorem 1. Hence, the series diverges. $\square$

Nonetheless, we can use the formula (18) in a different way to calculate the regular Abel function of $h(x) = \mathrm{e}^x - 1$. If we truncate v to N summands, denoted by v_N and $\alpha_N := s + v_N$, then Écalle showed (see [2], p. 78 ff., 95 ff.) that there are $2(m-1)$ different regular Abel functions $\alpha^{(j)}$, $1 \le j \le 2(m-1)$ ($\alpha^{(j)}$ is defined on the j-th petal — each petal touching the fixed point 0 — of the now-called Leau-Fatou flower, see [14]) given by:

$$\alpha^{(j)}(z) = \lim_{n\to\infty} \alpha_N^{(j)}(f^{[(-1)^{j+1}n]}(z)) - (-1)^{j+1}n, \quad N \in \mathbb{N} \tag{19}$$

where $\alpha_N^{(j)}(z)$ is $\alpha_N(z)$ with the logarithmic term $\log|z|$ (in (18)) replaced by $\log\left(ze^{-(\theta_0 + j\frac{\pi}{m-1})\mathrm{i}}\right)$ and θ_0 is the unique number in the interval $(-\frac{\pi}{m-1}, \frac{\pi}{m-1}]$ such that $h_m = |h_m|\, e^{-\mathrm{i}(m-1)\theta_0}$.

This applied to $h(z) = \mathrm{e}^z - 1$ where $m = 2$, $h_m = 1/2$, $\theta_0 = 0$, we get the two regular Abel functions

$$\alpha^{(1)}(z) = \lim_{n\to\infty} \frac{1}{3}\log(-h^{[n]}(z)) - \frac{2}{h^{[n]}(z)} + v_N(h^{[n]}(z)) - n, \quad \Re(z) < 0, \tag{20}$$

$$\alpha^{(2)}(z) = \lim_{n\to\infty} \frac{1}{3}\log(h^{[-n]}(z)) - \frac{2}{h^{[-n]}(z)} + v_N(h^{[-n]}(z)) + n, \quad \Re(z) > 0, \tag{21}$$

$$A^{(1)}(z) = \alpha^{(1)}\left(\frac{z}{\mathrm{e}} - 1\right), \Re(z) < \mathrm{e}, \tag{22}$$

$$A^{(2)}(z) = \alpha^{(2)}\left(\frac{z}{\mathrm{e}} - 1\right), \Re(z) > \mathrm{e}. \tag{23}$$

From these 4 methods we know that Lévy's formula (7), the Newton formula (9) and Écalle's method (20) and (21) calculate the *regular* iteration/Abel function. We show in the last part of this section that Walker's formula is equal to Écalle's formula and hence also computes the regular Abel function.

The application of Theorem 1.3.5 in [12] gives the following:

Preliminary 4 (Thron 1960, [16], Theorem 3.1)**.** *Let h be analytic at* 0 *with powerseries expansion of the following form:*

$$h(x) = x + h_m x^m + h_{m+1}x^{m+1} + \dots, \quad h_m < 0, m \ge 2,$$

then

$$\lim_{n\to\infty} n^{1/(m-1)} h^{[n]}(x) = (-h_m(m-1))^{-1/(m-1)}.$$

Now, about the functions $\alpha_{\mathrm{W}}^{(1)}$ and $\alpha_{\mathrm{W}}^{(2)}$ constructed by Walker, we have the following theorem:

Theorem 5. *Functions $\alpha_{\mathrm{W}}^{(1)}$ and $\alpha_{\mathrm{W}}^{(2)}$ given in* (12) *and* (13) *are the two regular Abel functions of $x \mapsto \mathrm{e}^x - 1$.*

Proof. We show that the difference of Walker's and Écalle's limit formulas is a constant. The differences are:

$$\delta_1(z) = \lim_{n\to\infty} \frac{1}{3}\log(-h^{[n]}(z)) + \frac{1}{3}\log(n) = \lim_{n\to\infty}\frac{1}{3}\log\left(-nh^{[n]}(z)\right), \quad z<0,$$
$$\delta_2(z) = \lim_{n\to\infty} \frac{1}{3}\log(h^{[-n]}(z)) + \frac{1}{3}\log(n) = \lim_{n\to\infty}\frac{1}{3}\log\left(nh^{[-n]}(z)\right), \quad z>0.$$

As $x \mapsto -h(-x)$ and $x \mapsto h^{-1}(x)$ for $h(x) = \mathrm{e}^x - 1$ is of the form required by Preliminary 4 with $m-1=1$ we see that each of $nh^{[n]}(z)$ and $nh^{[-n]}(z)$ converges to a constant independent on z. □

3. A new expansion of the super-exponentials

This section describes an evaluation of the two super-exponentials to base $b = \exp(1/\mathrm{e})$; this evaluation is faster and more precise than the one given in formula (9) contained in [2]. In particular, it allows us to plot the complex maps of these functions in real time. These maps are shown in Figure 2.

The base function $h(z) = \exp_b(z) = \exp(z/\mathrm{e})$ has many fixed points, but only one of them is real, namely, $z = \mathrm{e}$. The super-exponential is expected to approach this point asymptotically. Consider the expansion of the super-exponential $\tilde{F}$ in the following form:

$$\tilde{F}(z) = \mathrm{e}\cdot\left(1 - \frac{2}{z}\left(1 + \sum_{m=1}^{M} \frac{P_m\big(-\ln(\pm z)\big)}{(3z)^m} + \mathcal{O}\left(\frac{|\ln(z)|^{m+1}}{z^{m+1}}\right)\right)\right) \tag{24}$$

where

$$P_m(t) = \sum_{n=0}^{m} c_{n,m} t^n. \tag{25}$$

The substitution of (1) into equation

$$F(z+1) = \exp(F(z)/\mathrm{e}) \tag{26}$$

and the asymptotic analysis with small parameter $|1/z|$ determines the coefficients c in the polynomials (25). In particular,

$$P_1(t) = t, \tag{27}$$
$$P_2(t) = t^2 + t + 1/2, \tag{28}$$
$$P_3(t) = t^3 + \frac{5}{2}t^2 + \frac{5}{2}t + \frac{7}{10}, \tag{29}$$
$$P_4(t) = t^4 + \frac{13}{3}t^3 + \frac{45}{6}t^2 + \frac{53}{10}t + \frac{67}{60}, \tag{30}$$
$$P_5(t) = t^5 + \frac{77}{12}t^4 + \frac{101}{6}t^3 + \frac{83}{4}t^2 + \frac{653}{60}t + \frac{2701}{1680}. \tag{31}$$

The evaluation with 9 polynomials P gives an approximation of $F(z)$ with 15 decimal digits at $\Re(z) > 4$. For small values of z, the iterations of formula

$$F(z) = \ln(F(z+1))\ \mathrm{e} \tag{32}$$

can be used. With `complex<double>` precision, the resulting approximation returns an order of 14 correct decimal digits in the whole complex plane, except the singularities.

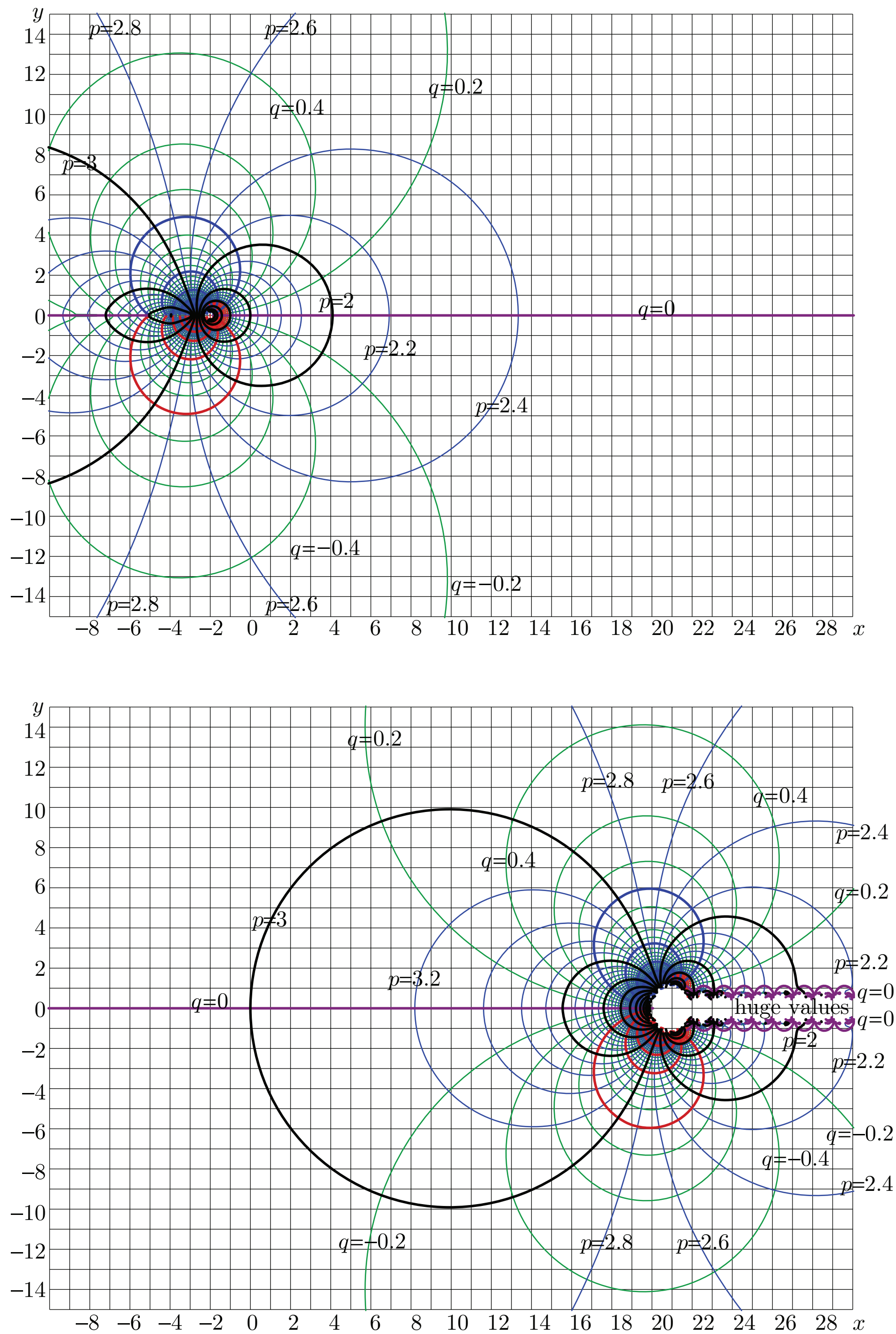


FIGURE 2. Map of $f = F_1(z)$, top, and $f = F_3(z)$, bottom, in the plane $z = x + \mathrm{i}y$. Levels $p = \Re(f) = \text{const}$ and $q = \Im(f) = \text{const}$ are shown; thick lines correspond to the integer values.

For the tetrational we choose the negative sign inside the logarithm $t = -\ln(-z)$. Then

$$F_1(z) = \tilde{F}(z + x_1) \tag{33}$$

where $x_1 \approx 2.798248154231454$ is the solution of the equation $\tilde{F}(x_1) = 1$ in such a way that $F_1(0) = 1$. For real values of the argument, this function is shown at the bottom of Figure 1. The complex map of this function is shown at the top of Figure 2.

The same expressions (24)–(31) can be used also for the above unbounded super-exponential with $t = -\ln(z)$. Then, the expression

$$F(z) = \exp(F(z-1)/\mathrm{e}) \tag{34}$$

allows the evaluation of the above unbounded super-exponential at small $|z|$. The specific super-exponential F_3 can be expressed as

$$F_3(z) = \tilde{F}(z + x_3) \tag{35}$$

where $x_3 \approx -20.28740458994004$ is the solution of equation $\tilde{F}(x_3) = 3$. Function F_3 by (35), (34), (1) is shown at the top of Figure 1 for real argument and in the bottom picture of Figure 2 for the complex values of its argument.

Due to the leading term in the asymptotic representation, at large values of $|z|$ (except the vicinity of the real axis), both functions $F_3(z)$ and $F_1(z)$ behave similar to the function $z \mapsto 1/z$.

F_3 is entire, and shows fast growth along the real axis. At large values of $|z|$, the function $F_1(z)$ approaches value e. Function $F_3(z)$ approaches value e for $|z| \to \infty$ except on the positive direction of the real axis; in this direction, this function shows "faster than any exponential" growth.

Although the series (24) seem to be calculable to an arbitrary number of terms, we present no rigorous proof of the existence of the solution; there is only computational evidence.

4. Numerics and behavior of the two super-logarithms

For the evaluation of the super-logarithms, the expression (18) is used. The truncation of the series v keeping the term of the 15th power was used to build up an approximation of $A^{(1)}$ that returns at least 15 decimal digits for $|z/\mathrm{e} - 1| < \frac{1}{2}$.

$$A^{(1)}(z) \quad = \quad \alpha(-\zeta) \approx \frac{\ln(\zeta)}{3} + \frac{2}{\zeta} + \sum_{n=1}^{15} c_n \zeta^n \tag{36}$$

where $\zeta = (\mathrm{e} - z)/\mathrm{e}$. For larger values, the representation

$$A^{(1)}(z) \quad = A^{(1)}\big(\exp(z/\mathrm{e})\big) + 1 \tag{37}$$

is iteratively used. This allows us to extend the approximation to a wide domain keeping an order of 14 correct decimal digits. Then

$$A_1(z) \quad = \quad A^{(1)}(z) - A^{(1)}(1) \quad \approx \quad A^{(1)}(z) - 3.029297214418 \tag{38}$$

is the regular Abel function with the additive constant chosen such that $A_1(1) = 0$. This function is shown in the top of Figure 3.

The function A_1 is periodic; the period is $T_1 = 2\pi \mathrm{e}\,\mathrm{i} \approx 17.079468445347134131\,\mathrm{i}$. For real values $z > \mathrm{e}$, the representation diverges, indicating a natural way to place the cut of the range of holomorphism. In the vicinity of this cut, the Abel-function

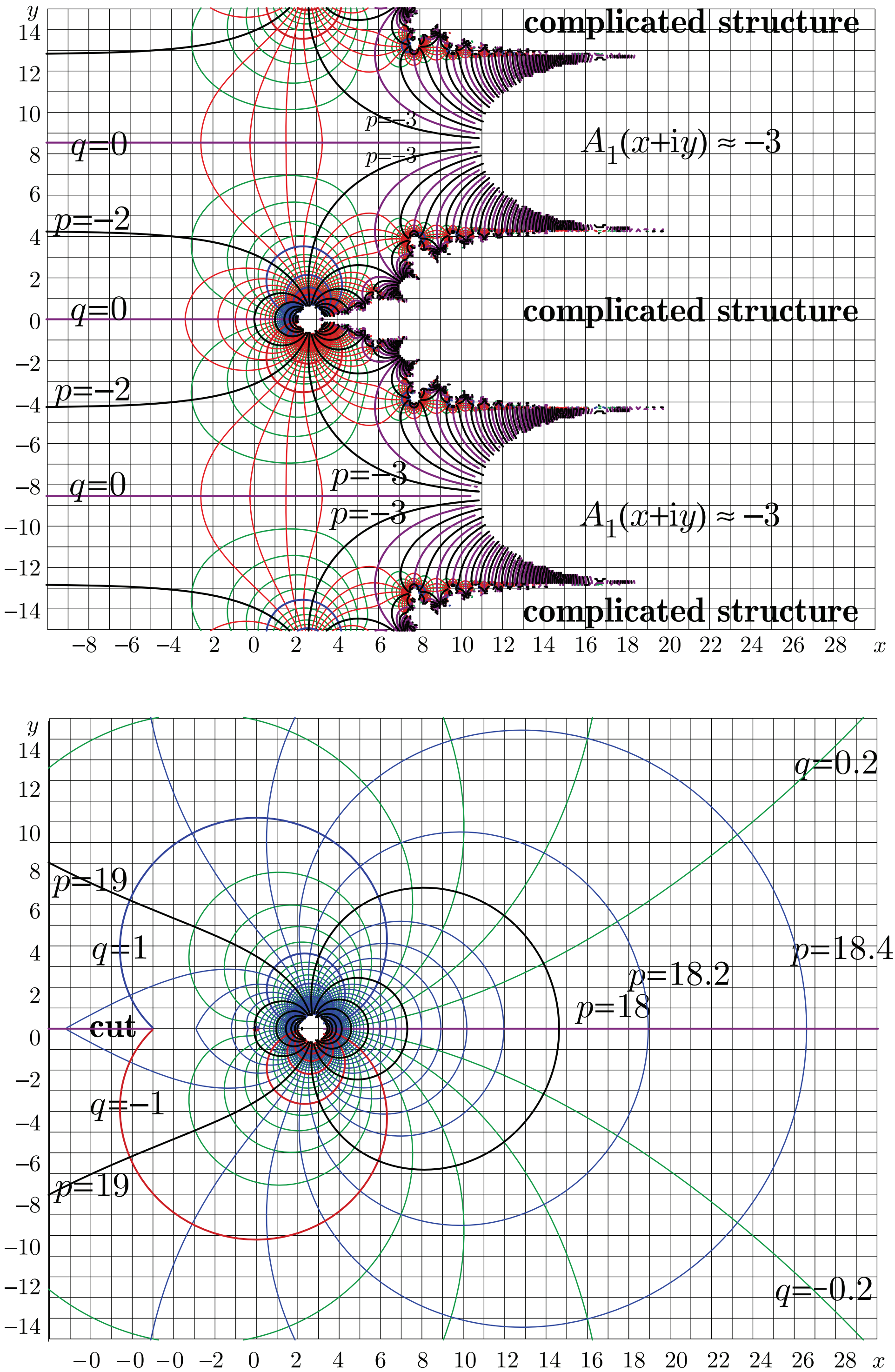


FIGURE 3. The Abel functions $f = A_1(z)$, top, and $f = A_3(z)$, bottom, in the same notations as in Figure 2.

shows complicated, fractal-like behavior: the self-similar structures reproduce along the range with high density of levels of constant real or imaginary part of A_1.

At the left-hand side of the picture, $A_1(z)$ approaches asymptotic value -2 as $\Re(z) \to -\infty$. Along the strips in vicinity $\Im(z) = \Im(T_1 + \pi\mathrm{i}(2n+1))$, $n \in$ integers, as $\Re(z) \to +\infty$, this function approaches another limiting value: $A_1(z) \to -3$. The transfer from the asymptotic value -2 to the asymptotic value -3 corresponds to the transition from the singularity at $z = -2$ to the singularity $z = -3$ of function $F_1(z)$ in the top picture of Figure 2.

The second super-logarithm $A^{(2)}$ has also good approximation for small values of $|\zeta|$ for the same $\zeta = (\mathrm{e} - z)/\mathrm{e}$:

$$A^{(2)}(z) \approx \frac{\ln(-\zeta)}{3} + \frac{2}{\zeta} + \sum_{n=1}^{16} c_n \zeta^n \tag{39}$$

with the same coefficients c, as in the case of $A^{(1)}$. (The only difference is the opposite sign in the argument of the logarithm.) For large values, an extension to a wide range in the complex plane can be similarly realized with

$$A^{(2)}(z) = A^{(2)}\big(\log(z)\ \mathrm{e}\big) - 1. \tag{40}$$

The resulting function

$$A_3(z) = A^{(2)}(z) - A^{(2)}(3) \approx A^{(2)}(z) + 20.0563555297533789 \tag{41}$$

is plotted in the right bottom part of Figure 3. Function $A_3(z)$ is not periodic, and has a cut from the branch-point $z = \mathrm{e}$ to the negative direction of the real axis. In the positive direction of the real axis, it grows to infinity, and this growth is very slow (slower than any finite combination of logarithms).

The asymptotic representation for the Abel functions A_1 and A_3 can be inverted, using various combinations of $A - \mathrm{e}$ and $\ln\big(\pm(A - \mathrm{e})\big)$ as a small parameter. A different small parameter allows different representations for the super-exponentials F_1 and F; it seems many of them give comparable speed and comparable prevision; at least they do not add many errors to the rounding errors at the complex < double > implementation. The asymptotics of the previous section seems to be the fastest, although the careful comparison of efficiency of various asymptotic formulas may be the subject for future investigation.

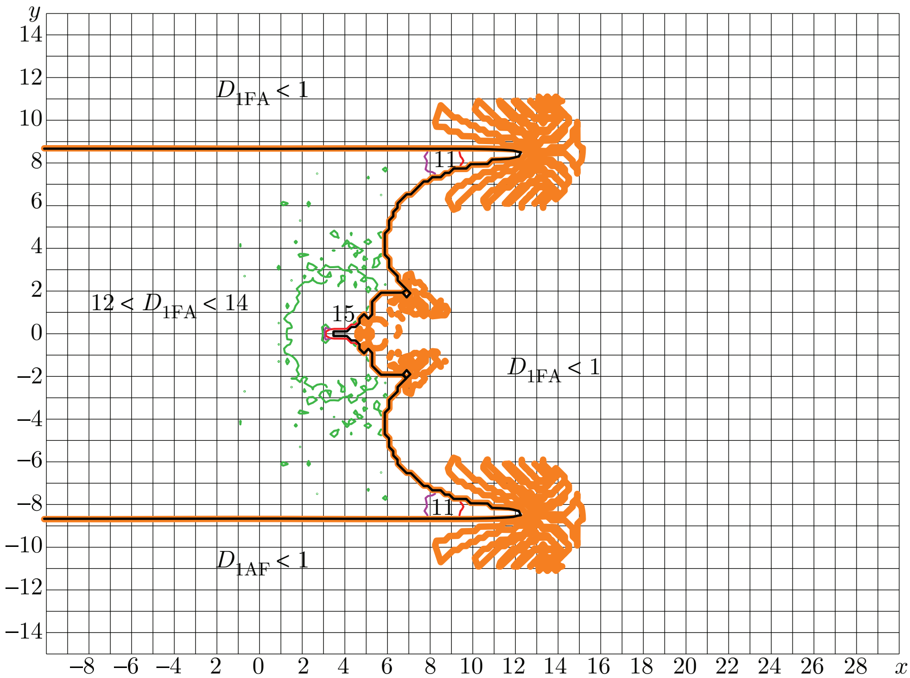
For the plotting of Figure 3, the algorithms for A_1 and A_3 were implemented in C++ with complex < double > arithmetics. In order to verify the consistency of these algorithms to those for F_1 and F_3, the following agreements are considered:

$$D_{1\mathrm{AF}}(z) = \lg\left|\frac{A_1(F_1(z)) + z}{A_1(F_1(z)) - z}\right|, \qquad D_{1\mathrm{FA}}(z) = \lg\left|\frac{F_1(A_1(z)) + z}{F_1(A_1(z)) - z}\right|, \tag{42}$$

$$D_{3\mathrm{AF}}(z) = \lg\left|\frac{A_3(F_3(z)) + z}{A_3(F_3(z)) - z}\right|, \qquad D_{3\mathrm{FA}}(z) = \lg\left|\frac{F_3(A_3(z)) + z}{F_3(A_3(z)) - z}\right|. \tag{43}$$

Two of them, namely, $D = D_{1\mathrm{AF}}(z)$ and $D = D_{3\mathrm{FA}}(z)$ are shown in Figure 4 with contours $D = \mathrm{const}$. As for $D_{1\mathrm{FA}}(z)$ and $D_{3\mathrm{AF}}(z)$, they remain an order of 14 in the whole range of such a picture, so they are not presented here.

In Figure 4, symbol "15" indicates the region where $D > 14$, and symbol "11" indicates the ranges where $10 < D < 12$.



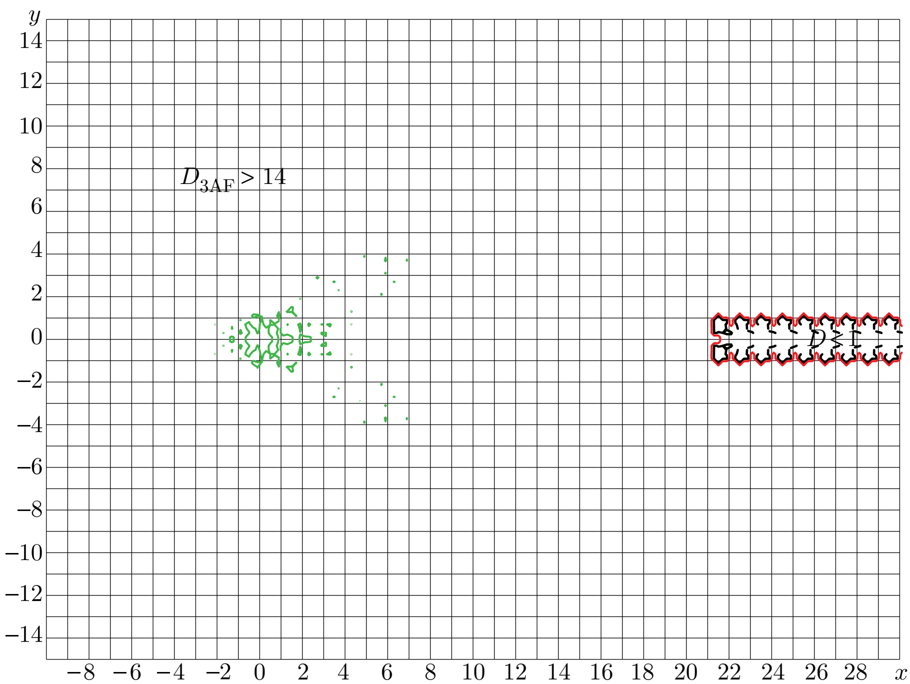


FIGURE 4. The agreements $D = D_{1FA}(z)$ and $D = D_{3AF}(z)$ and in the complex plane $z = x+iy$; level $D=1$ is shown with thick lines; the integer values are shown with thin lines; levels $D=2$ is shown with black thick lines, levels $D = 1, 2, 10, 12, 14$ are seen; symbols "15" and "11" indicate the ranges where $D > 14$ and $10 > D > 12$.

The top picture in Figure 4 shows, that in the central part, the implementations of functions F_1 and $A_1 = F_1^{-1}$ are consistent within at least 14 decimal digits. In the right-hand side, the branches of functions F_1 and A_1 do not match, and the agreement is poor.

The good agreement indicates, that the algorithms above work close to the best precision achievable with the `complex<double>` variables. However, the range of validity of relation $F(F^{-1}(z)) = z$ is limited by the cut lines of the functions F and F^{-1}. In such a way, the numerical tests confirm the efficient C++ implementation of the super-exponentials F_1 and F_3 and the corresponding Abel-exponentials A_1 and A_3.

5. Non-integer iteration

Each of the pairs (F_1, A_1) and (F_3, A_3) can be used to construct the regular iteration of the exponential to base $b = \exp(1/\mathrm{e})$:

$$\exp_{b,1}^{[c]}(z) = F_1\big(c + A_1(z)\big), \tag{44}$$

$$\exp_{b,3}^{[c]}(z) = F_3\big(c + A_3(z)\big). \tag{45}$$

These functions are shown in Figure 5 for $c = 1/2$. For comparison, the function $y = \exp_b(z)$ is plotted with a thin curve. Visually, the thick solid curve looks like a continuation of the dashed curve. However, analytic continuation is not possible because $z = \mathrm{e}$ is a branch point of $A_1(z)$. Similar visual effects are discussed in [9] for the case $b = \sqrt{2}$.

In order to see that $\exp_{b,1}^{[c]}(z)$ and $\exp_{b,3}^{[c]}(z)$, at least for $b = \exp(1/\mathrm{e})$ and $c = 1/2$, are pretty different functions, the difference of the "continuations" of these two functions, i.e.,

$$d_{\mathrm{q}13}(x) = \exp_{b,1}^{[1/2]}(x + \mathrm{i}o) - \exp_{b,3}^{[1/2]}(x + \mathrm{i}o) \tag{46}$$

in the vicinity of $x = \mathrm{e}$ is shown at the bottom picture of Figure 6.

The complex map of the two square roots of the exponential to base $\exp(1/\mathrm{e})$ are plotted in Figure 6 in the same notations, as in Figures 2 and 3. The additional levels $p = 0.692907175521155$ and $q = \pm 8.53$ are plotted in order to reveal the behavior of the $\exp_{b,1}^{[0.5]}(z)$ in the strips along the cut lines $\Re(z) > 0.7$, $\Im(z) \approx \pm 8.5$. The comparison of the pictures in Figure 6 show how different the functions $\exp_{b,1}^{[0.5]}$ and $\exp_{b,3}^{[0.5]}$ are when considered away from the real axis.

For the functions $\exp_{b,1}^{[1/2]}(z)$ and $\exp_{b,3}^{[1/2]}(z)$, in wide ranges of z, the relations

$$\exp_{b,1}^{[1/2]}\left(\exp_{b,1}^{[1/2]}(z)\right) = z, \tag{47}$$

$$\exp_{b,3}^{[1/2]}\left(\exp_{b,3}^{[1/2]}(z)\right) = z, \tag{48}$$

hold. As in the case of the square root of the logistic operator [11], the ranges of validity of these equations do not cover the whole complex plane, and they are

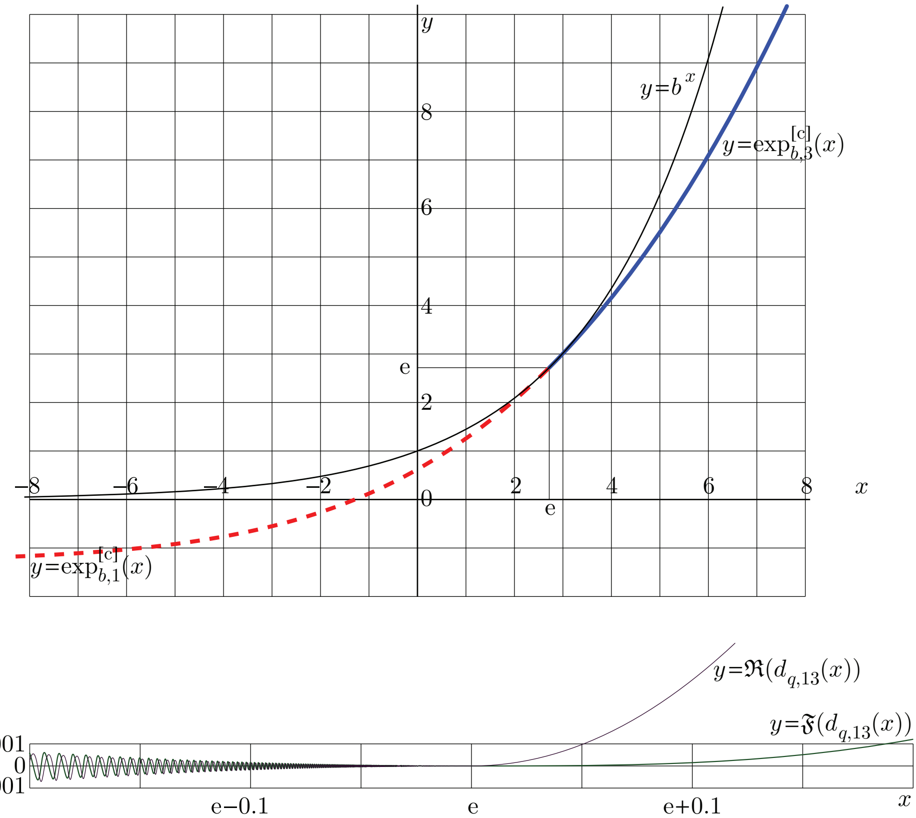


FIGURE 5. Behavior of the two square roots ($c\!=\!\frac{1}{2}$) of the exponential to base $b\!=\!\exp(1/\mathrm{e})$. Top picture: Dashed: $y\!=\!\exp_{b,1}^{[1/2]}(x)$ by (44); Thick solid: $y\!=\!\exp_{b,3}^{[1/2]}(x)$ by (45); Thin solid: $y\!=\!\exp_b(x)\!=\!b^x$. Bottom picture: $y\!=\!\Re\big(d_{\mathrm{q}13}(x)\big)$, thick line, and $y\!=\!\Im\big(d_{\mathrm{q}13}(x)\big)$, thin line, in the vicinity of $x\!=\!\mathrm{e}$ by equation (46).

different. In order to show these ranges, the agreements

$$D_{\mathrm{q}1}(z) = \lg \left| \frac{\exp_{b,1}^{[1/2]}\Big(\exp_{b,1}^{[1/2]}(z)\Big) + \exp_b(z)}{\exp_{b,1}^{[1/2]}\Big(\exp_{b,1}^{[1/2]}(z)\Big) - \exp_b(z)} \right|, \tag{49}$$

$$D_{\mathrm{q}3}(z) = \lg \left| \frac{\exp_{b,3}^{[1/2]}\Big(\exp_{b,3}^{[1/2]}(z)\Big) + \exp_b(z)}{\exp_{b,3}^{[1/2]}\Big(\exp_{b,3}^{[1/2]}(z)\Big) - \exp_b(z)} \right|, \tag{50}$$

are shown in Figure 7. In particular, $\exp_{b,3}^{[1/2]}(x)$ can be considered as the "true" square root of the exponential to base $b\!=\!\exp(1/\mathrm{e})$ for $x\!<\!\mathrm{e}$, while $\exp_{b,3}^{[1/2]}(x)$ can be considered as the "true" square root at $x\!>\!\mathrm{e}$; but these square roots cannot be combined into the same holomorphic function.

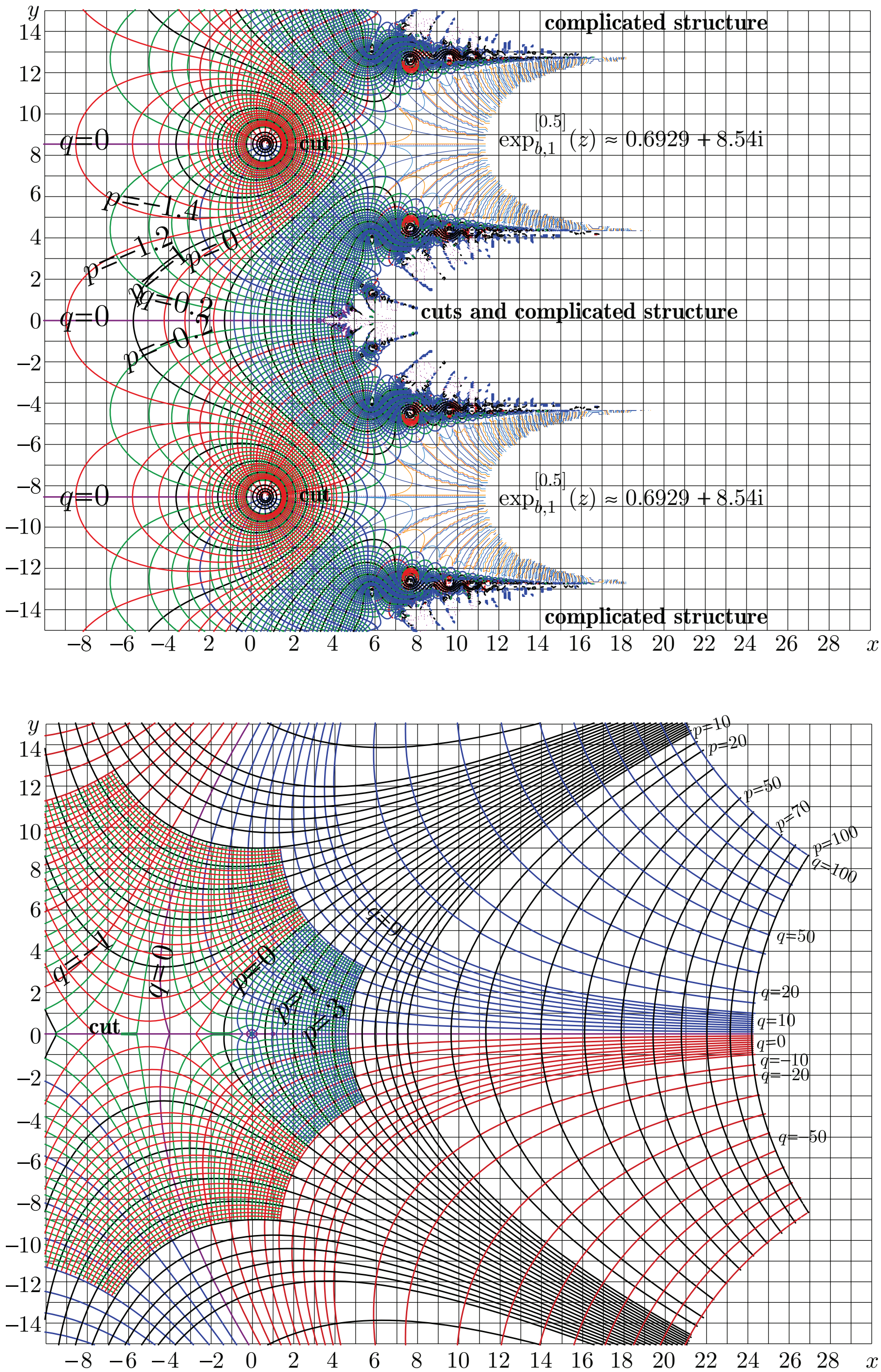


FIGURE 6. Two different "square roots" of the exponential to base exp(1/e) by equations (44) and (45) in the same notations as in Figures 2 and 3.

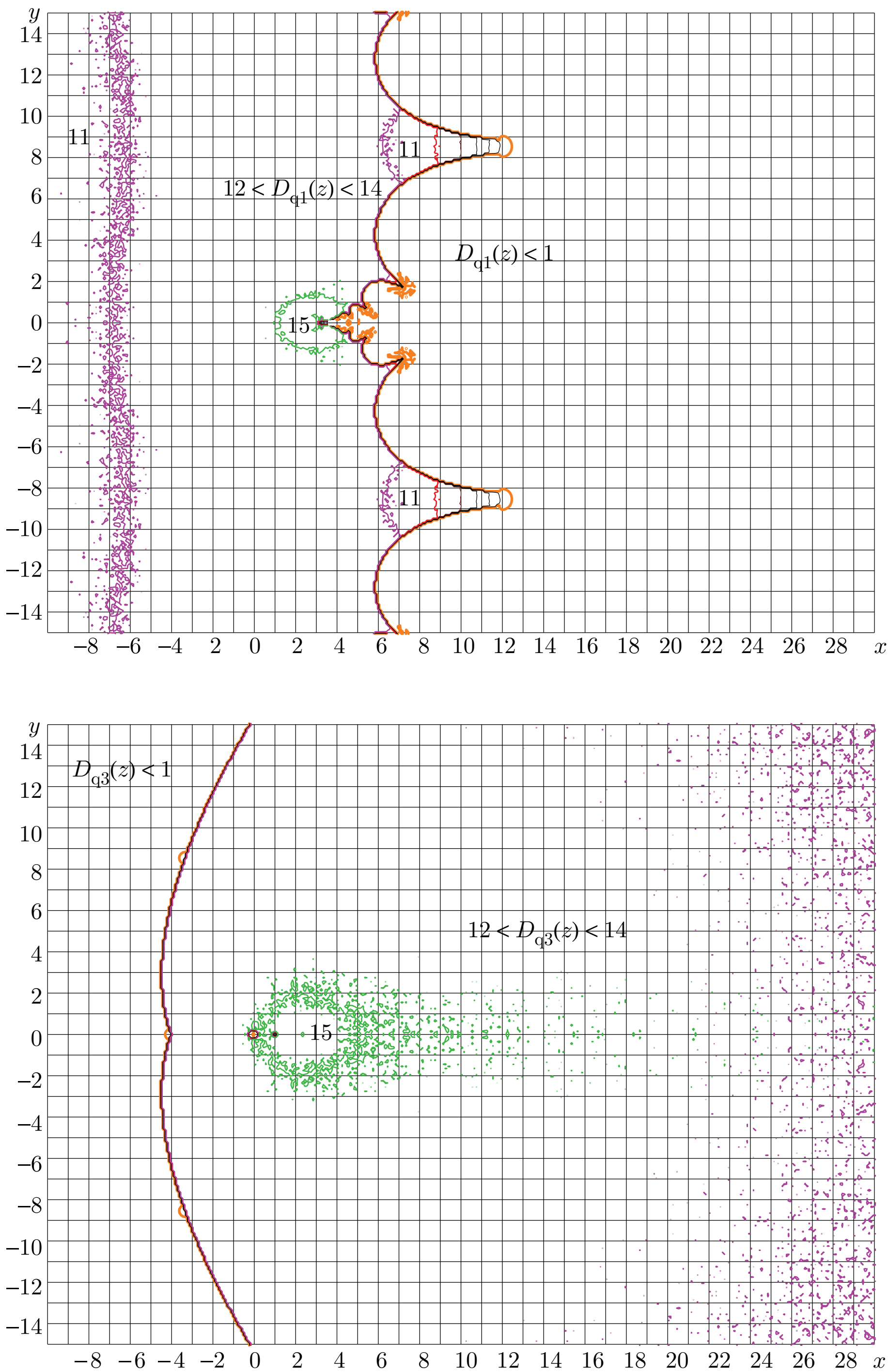


FIGURE 7. The agreements $D_{\mathrm{q1}}(z)$ and $D_{\mathrm{q3}}(z)$ by equations (49) and (50) in the complex plane $z = x + \mathrm{i}y$.

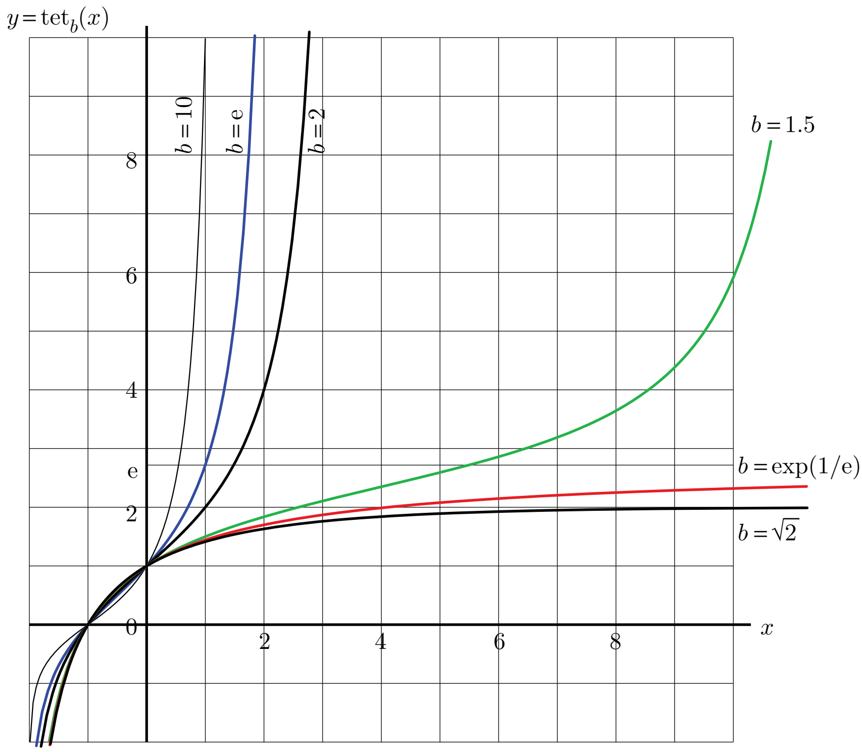


FIGURE 8. Tetrationals to base $b=10$, $b=\mathrm{e}\approx 2.71$, $b=2$, $b=1.5$, $b=\exp(1/\mathrm{e})\approx 1.44$ (the same curve as F_1 in Figure 1), and $b=\sqrt{2}\approx 1.41$ versus real argument.

The fractional iterations by (44), (45) can be evaluated for complex values of z and even for complex values of c; but only at integer values of c can these two functions be considered as holomorphic extensions of each other. The fractional iteration provides a smooth (holomorphic) transition from the exponential at $c=1$ to the logarithm at $c=-1$, passing through the "square root" of the exponential at $x=1/2$, the identity function at $c=0$ and the "square root" of the logarithm at $c=-1/2$. In a similar way, the complex iterations of a function can be considered.

The non-integer iteration of the exponentials provides a set of functions that grow up faster than any polynomial but slower than any exponential. Such functions may find applications in various areas of physics and technology. Similar non-integer iterations for other functions (including the exponentials of different bases and factorial) were discussed recently [7, 9, 10, 11], but the peculiarity of the fixed points of the exponential at the base $b=\exp(1/\mathrm{e})$ required the special consideration above.

6. Comparison of the tetrationals to different bases

In this section, the tetrational F_1 to base $\exp(1/\mathrm{e})$ is compared to tetrationals to various bases. In Figure 8, the tetrational tet_b versus real argument is shown for $b=10$, $b=\mathrm{e}\approx 2.71$, $b=2$, $b=1.5$, $b=\exp(1/\mathrm{e})\approx 1.44$ and $b=\sqrt{2}\approx 1.41$. The

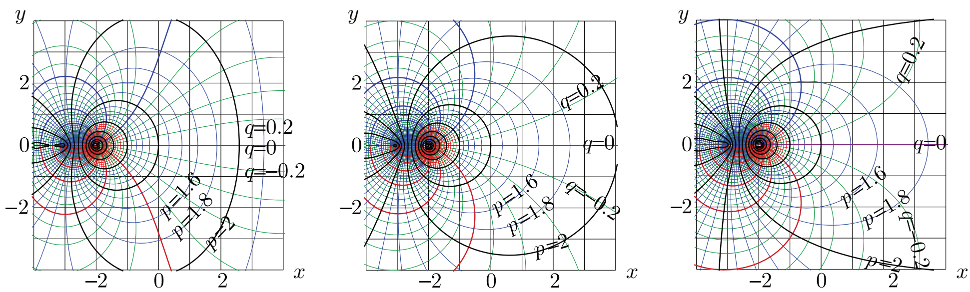


FIGURE 9. Tetrationals $\text{tet}_b(x+\text{i}y) = p+\text{i}q$ for various b, from left to right: $b=1.5$, $b=\exp(1/\text{e})\approx 1.44$ (the central part of map of F_1 at the top picture in Figure 2), and $b=\sqrt{2}\approx 1.41$ in the x, y plane.

functions for $b > \exp(1/\text{e})$ are evaluated using the Cauchy algorithm described in [7]. For $b<\exp(1/\text{e})$, the regular iteration described in [9] is used. For $b=\exp(1/\text{e})$, the tetrational is just F_1 shown also in Figure 1).

At moderate values of argument or order of unity or smaller, the curves for $b=1.5$, $b=\exp(1/\text{e})$ and $b=\sqrt{2}$ are very close. In order to see the difference, the complex maps are shown in Figure 9 for $b=1.5$, left, for $b=\exp(1/\text{e})$, central, and for $b=\sqrt{2}$, right. The central picture of Figure 9 is just a zoom-in from the central part of the top picture in Figure 2. The efficient algorithms of the computation allow us to plot all the figures with some reserve of resolution; at the online version, they still can be zoomed-in.

Figure 9 indicates no qualitative change of the tetrational at small variation of the base in the vicinity of the value $b = \exp(1/\text{e})$. In particular, within the loop $p=\Re\big(\text{tet}_b(z)\big)=1$ (this loop goes through the origin of coordinates in all three pictures), the zooming-in of the central parts of the pictures in Figure 9 is necessary to see the difference.

7. Comparison of the non-integer iterates of the exponential to different bases

The non-integer iterates $\exp_b^{[c]}(x)$ are shown in Figure 10 versus x for $b = \text{e}$, $b=\exp(1/\text{e})$, $b=\sqrt{2}$ and various values of c. At $1 < b \le \exp(1/\text{e})$ there are two c-iterates of the exponential. The one valid for x below the upper fixed point is shown with a dashed line; the one valid for x above the lower fixed point is drawn with a solid line. The curves for $b=\exp(1/\text{e})$, $c=1/2$ and $c=1$ are the same as in the top picture of Figure 5. For the evaluation of the iterated exponential to base $b=\text{e}$, the approximation by [8] is used (although the direct method by [7] could be used instead); the two iterated exponentials to base $b=\sqrt{2}$ were evaluated with the algorithms described in [9]. In the last case, in the range $2<x<4$ both of the iterated exponentials are valid, and the deviation between the two iterated exponentials is of order of 10^{-24}; however, even in this case they are not holomorphic extensions of each other. (Being plotted on a paper, the distance between the two curves for the same c is small not only in comparison to the size of an atom, but also small being compared to the size of atomic nuclei.)

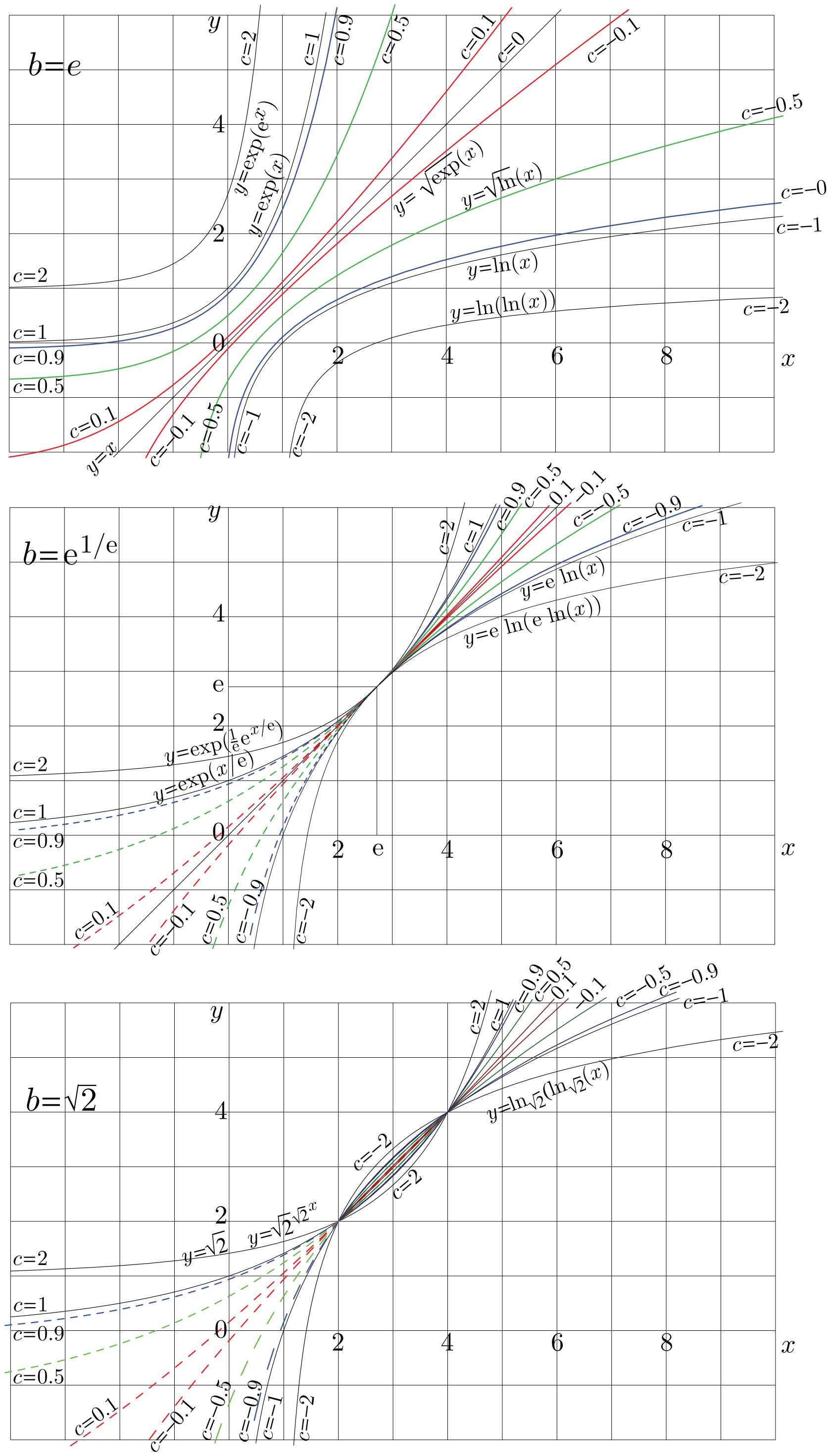


FIGURE 10. The iterated exponentials $y = \exp_b^{[c]}(x)$ versus real x for various real c, for bases $b = \mathrm{e}$, $b = \exp(1/\mathrm{e})$ and $b = \sqrt{2}$.

8. Conclusion and prospects

We present some theory of regular iteration and apply it to the case $f(z) = \mathrm{e}^{z/\mathrm{e}}$. We extract a quite efficient algorithm to calculate the two super-logarithms to base $\mathrm{e}^{1/\mathrm{e}}$ compared with various other (standard and separate) methods. We suggest an efficient new non-polynomial approximation for the two super-exponentials to base $\mathrm{e}^{1/\mathrm{e}}$.

One of the two super-exponentials, the tetrational $F_1 = \mathrm{tet}_{\exp(1/\mathrm{e})}$, is holomorphic in the range $\mathbb{C} \setminus \{x \in \mathbb{R} : x \le -2\}$ and strictly increasing along the real axis > -2. It asymptotically approaches the limiting value e. The function approaches the same value e also in any other directions, i.e., at large values of $|z|$. The jump at the cut $z \le -2$ reduces to zero, as $|z| \to \infty$.

The other super-exponential F_3 is an entire function. Along the real axis it is strictly increasing from the limiting value e at $-\infty$ to infinity, growing faster than any exponential. Outside the positive part of the real axis, $F_3(z)$ approaches e at $|z| \to \infty$ in a similar way as F_1 does.

Efficient calculation algorithms and portraits for bases $b > \mathrm{e}^{1/\mathrm{e}}$ and $1 < b < \mathrm{e}^{1/\mathrm{e}}$ were already provided in [7, 9]; so the whole range $b > 1$ is now covered. The plots of the tetrational tet_b for $b = \mathrm{e}^{1/\mathrm{e}} \approx 1.44$ look similar to those for $b = 1.5$ and those for $b = \sqrt{2} \approx 1.41$; one may expect that at any fixed value of z from some range, the tetrational $\mathrm{tet}_b(z)$ is a continuous function of b at least for $b > 1$. This raises the following questions:

Are the holomorphic tetrationals constructed in [9], here and in [17] (generalization of [6] which is conjectured to be the super-exponential in [7]) — which together cover the base range $(1, \infty)$ — analytic as a function of the base b, particularly in the point $b = \mathrm{e}^{1/\mathrm{e}}$? If so, what is the range of holomorphism? If not: can one obtain an operation $(b, z) \mapsto \mathrm{tet}_b(z)$ defined for b in a vicinity of $\mathrm{e}^{1/\mathrm{e}}$ such that for each b the function $z \mapsto \mathrm{tet}_b(z)$ is a real-analytic tetrational on $(-2, \infty)$ and the function $b \mapsto \mathrm{tet}_b(z)$ is holomorphic for each z?

There is a similar bifurcation base $b \approx 1.6353$ for the tetrational $\mathrm{tet}_b(z)$ as the bifurcation base $e^{1/e}$ is for the exponential (i.e. where the two fixed points change into no fixed point). One could apply the same methods we used to obtain the super-exponentials to also obtain super-tetrationals/pentationals.

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