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Entire Function with Logarithmic Asymptotic

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Abstract

For the transfer function T(z)=z+exp(z), the entire superfunction f is constructed as solution of the transfer equation f(z+1)=T(f(z)). The efficient algorithm for evaluation of Superfunction f is suggested. Its logarithmic asymptotic behaviour is detected. The application for emulation of the electric field of a charged wire in empty space is discussed.

Mathematics Subject Classification: 30D15, 30C30

Keywords: Entire function, Superfunction, Abelfunction

1 Introduction

Usually, a holomorphic function with logarithmic asymptotic has some singularity (at least logarithmic) in the complex plane. Such singularity is absent for function, referred below as SuTra. Plot $y=\mathrm{SuTra}(x)$ is shown in figure 1 with thick curve. For comparison, the thin curve shows $y=-\ln(-x)$. Complex map of function SuTra is shown in figure 2. This article describes construction and evaluation of function SuTra.

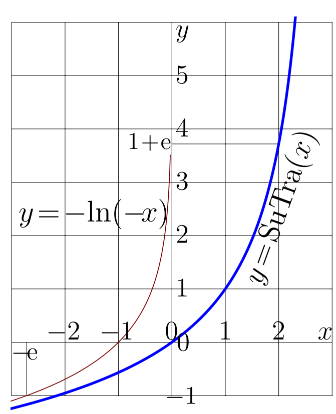

Figure 1: $y=-\ln(-x)$ and $y=\mathrm{SuTra}(x)$

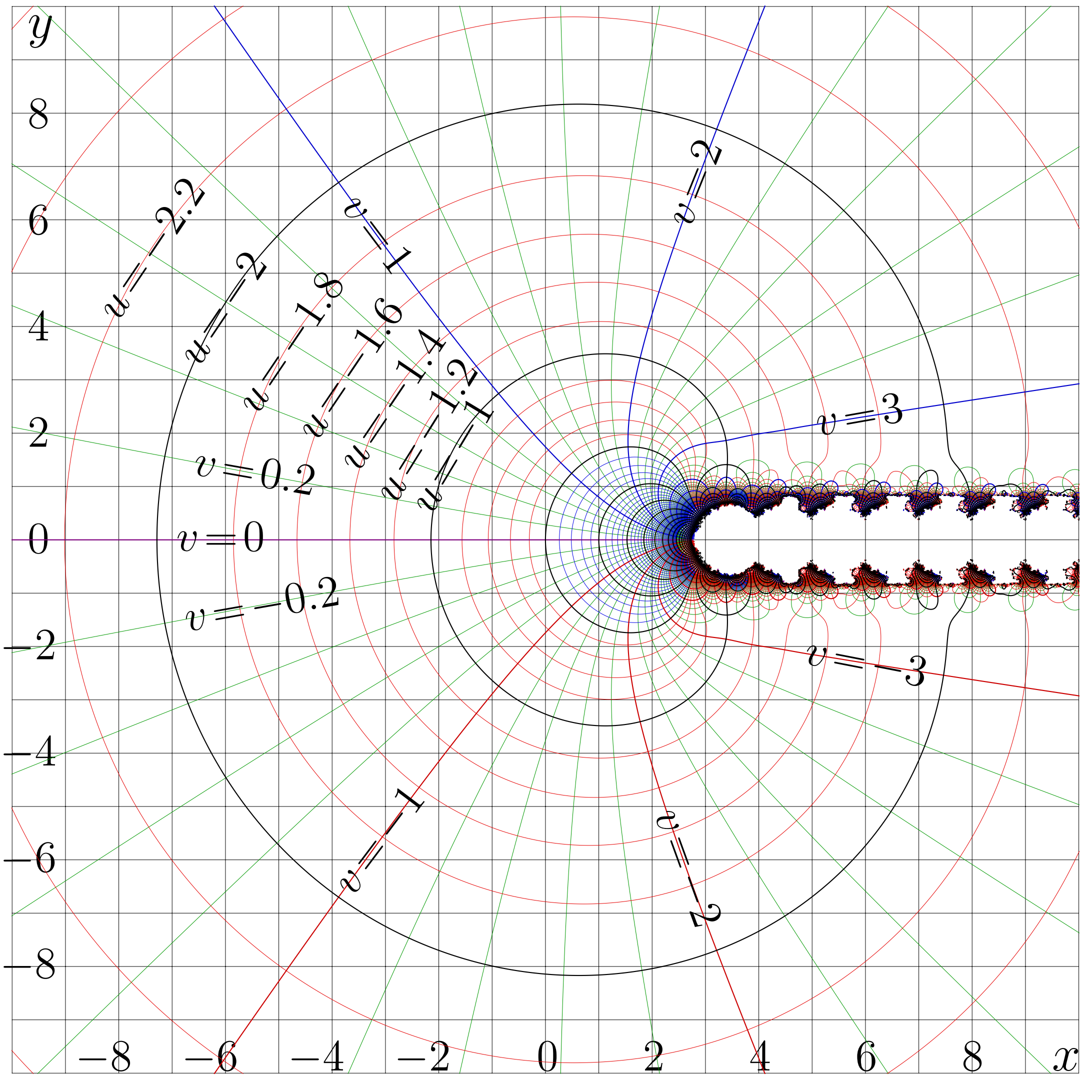

Figure 2: Map of function SuTra; $u+\mathrm{i}v = \mathrm{SuTra}(x+\mathrm{i}y)$

Function SuTra is superfunction [1, 2, 3, 4] for the elementary function

$$\mathrm{tra}(z) = z + \exp(z) \tag{1}$$

id est, it is solution SuTra of the transfer equation

$$\mathrm{SuTra}(z+1) = \mathrm{tra}\Big(\mathrm{SuTra}(z)\Big) \tag{2}$$

The additional condition

$$\mathrm{SuTra}(0) = 0 \tag{3}$$

is applied; then, the graphic at Figure 1 passes through points (0,0), (1,1) and (2,1+e); these values are seen also in the map at figure 2.

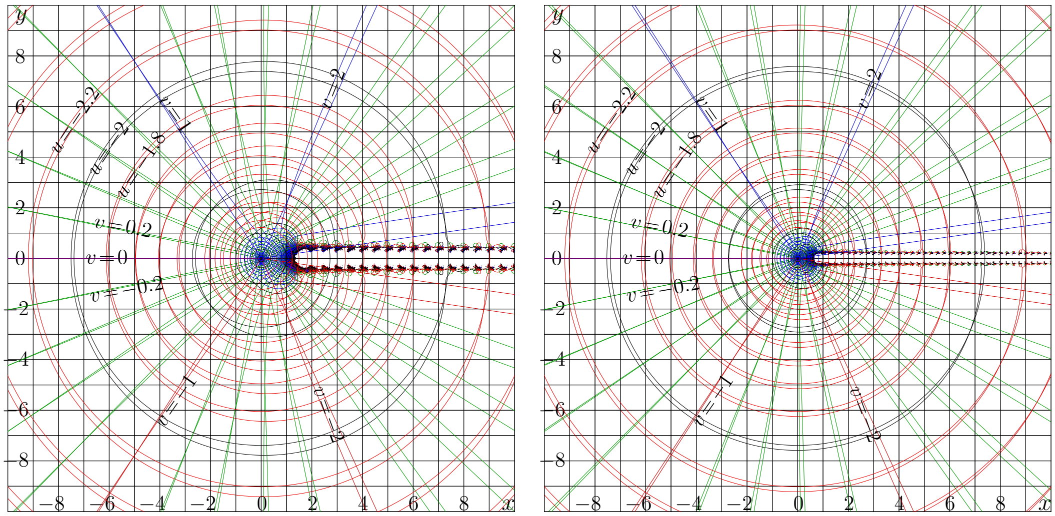

Figure 3: Maps of functions $f_2(z)$ by (4), left, and $f_4(z)$ by (5), right, overlapped with map of $f_\infty(z)$ by (6) in the same notations as in figure 2

In order to see the logarithmic asymptotic of function SuTra in the complex plane, figure 3 shows the maps of functions

$$f_2(z) = \mathrm{SuTra}(2z) + \ln(2) \tag{4}$$

$$f_4(z) = \mathrm{SuTra}(4z) + \ln(4) \tag{5}$$

For comparison, the map of function

$$f_\infty(z) = -\ln(-z) \tag{6}$$

is shown in the same pictures. The straight lines and ideal circles correspond to function f_∞; other curves refer to functions f_2 and f_4. In the right hand side picture, the deviation is so small, that it is difficult to guess, which line corresponds to the elementary function f_∞ by (6) and which refer to function f_4 by (5).

The logarithmic function can be approximated with entire function SuTra in wide range of values; for the most of the complex plane (except $z \le 0$)

$$\ln(z) = \lim_{S\to\infty} \Big(-\mathrm{SuTra}(-Sz) - \ln(S) \Big) \tag{7}$$

In such a way, in the whole complex plane (except zero and negative part of the real axis), logarithm appears as limit of the entire function. Representation (7) justifies the title of this article. The following sections describe the construction and evaluation of function SuTra:

Section 2 provides some preliminary notes and definitions about the superfunctions, collecting formulas and notations from the literature cited.

Section 3 specifies the asymptotic behaviour of function SuTra and provides its definition.

Section 5 analyses the range of validity of the primary approximation of function SuTra and describes its numerical implementation.

Section 5 discuss the relation of function SuTra to other functions

Section 6 concludes the article.

2 Preliminary Notes

This section collect some definitions about superfunctions and some basic formulas. Term "superfunction" seems to be most convenient among variety of terminologies used in the literature [1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16] for a solution of the transfer equation (8). Formulas of this section are not new, but I hope, they release the reader from the need to browse previous publications on the topic mentioned above.

Definition 2.1 *Let* $C \subset \mathbb{C} : z \in C \to z+1 \in C$.
Let $T : D \mapsto D$ *be holomorphic at some* $D \subset \mathbb{C}$. *Let* C *and* D *be connected manifolds. Then, holomorphic function* $F : C \mapsto D$ *is called* **superfunction** *of function* T*, iff for* $z \in C$*, the equation below holds*

$$T\Big(F(z)\Big) = F(z+1) \tag{8}$$

In this case, function T *is qualified as* **transfer function** *for function* F*;*
and equation 8 is called **transfer equation***.*
If, in addition to (8), for $a \in C$ *and* $b \in D$*, the relation* $F(a)=b$ *holds, then,* F *is "*$a \mapsto b$ *superfunction of* T*".*

Several examples of super functions are collected in the Tables of superfunctions [1, 23]. Super function is indicated adding prefix "Super" or "Su" to the name of its transfer function. For example, any superfunction of exponential is called "superexponential". Superexponential with specific value at zero and specific asymptotic behaviour in the complex plane is called "tetration", and efficient algorithms for the precise evaluation are available [10, 11, 12, 13].

Definition 2.2 *Let* $F : C \mapsto C$ *be superfunction for function* T*.*
Let $c \subset \mathbb{C} : z \in c \to T(z) \in c$ *, and let* $G : c \mapsto c$ *be holomorphic function such that* $\forall z \in c$ *the equation below holds.*

$$F(G(z)) = z \tag{9}$$

Then, function G *is called* **abelfunction** *of function* T*.*

In this way, the abelfunction G appears as inverse of the superfunction F, id est, $G = F^{-1}$. The abelfunction satisfied the special Abel equation below:

Theorem 2.3 *Let G be abelfunction of T. Then, there exists $c \in \mathbb{C}$ such that for all $z \in c$, the following equation holds*

$$G(T(z)) = G(z) + 1 \tag{10}$$

Proof Begin with the transfer equation (8). Replace z to $G(z)$. This gives

$$T(F(G(z))) = F(G(z) + 1) \tag{11}$$

Apply equation (9) to the left hand side of equation (11). The result is

$$T(z) = F(G(z) + 1) \tag{12}$$

Apply function G to both sides of equation (12). This gives

$$G(T(z)) = G(F(G(z) + 1)) \tag{13}$$

In the range of isomorphism of $F \leftrightarrow G$, function F is inverse of function G. For this range, the Abel equation (10) holds.
End of Proof

In the literature, various equations are called "The Abel equation". In order to avoid confusions, in this article the following definition is used:

Definition 2.4 *Let T be given transfer function. Then, equation (10) for the abelfunction G is called* **Abel equation**.

Properties of superfunction F and abelfunction G allow to define the nth iterate of the transfer function T in the following way:

Definition 2.5 *Let F be superfunction of function T, and let G be corresponding abelfunction. Then, the nth iterate of T is the following function:*

$$T^n(z) = F(n + G(z)) \tag{14}$$

In this article, the number in superscript after the name of function indicates number of its iterate, and never indicates argument of exponentiation to apply after the evaluation of function; if exponentiation, the argument is written as a superscript after the end of specification of the base of the exponent. For example, in these notations, $\sin^2(z) = \sin(\sin(z))$, but never $\sin(z)^2$. The similar notation is used in Quantum mechanics; where $\hat{P}^2\psi$ or $\hat{P}^2(\psi)$ meas $P(P(\psi))$, but never $(P(\psi))^2$. Notation T^n for the nth iteration of function T is not new; Walter Bergwejer used this notation in century 20 [9].

Holomorphism of superfunction F determines that, at fixed z from the range of definition of G, the iterate T^n is holomorphic function with respect to the number n of iterations. Definitions of superfunction and the abelfunction provide that.

Theorem 2.6 *For given transfer function T and its superfunction F and corresponding abelfunction G, there exist some open set of ranges of complex numbers m, n, z, where the following equation holds:*

$$T^m(T^n(z)) = T^{m+n}(z) \tag{15}$$

Proof Using definition of iteration,express the left hand side of equation (15) in terms of superfunciton F and abelfunction G. For $G(z+n)$ that belongs to the range of isomorphism of functions F and G, the relation $G(F(n + G(z))) = n + G(z)$ can be used. Then, these F and G can be canceled, providing expression $F(m+n+G(z))$ in the left hand side of (15). Then, the definition (14) converts the equation to identity.

End of proof.

In such a way, the theorem above states the group properties of the iteration. In particular, this group property can be used for integer values of m and n. The group property (15) justifies the use of superscripts to indicate the iterate; the functions can be combined, as if they would be powers of some operators, as it takes place in the Quantum Mechanics. (This analogy does not go far: for non-linear holomorphic functions, there is no distributive law, e.g., $P(\varphi+\psi) = P\varphi + P\psi$, postulated for linear operators in Quantum Mechanics.)

Ability to evaluate non-integer iterates of functions greatly extends the set of functions that can be used to approximate the physical dependences. This is expected to find application in physics and other sciences.

The iterates above are not unique, as not unique are pairs (superfunction, abelfunction). For this reason, the definition of iterates indicates, that the superfunction F and the abelfunciton G are given. In many cases, the criteria of simplicity allow to choose the most "physical" superfunction among various solutions of the transfer equation (8) [16]. For the transfer function tra by (1), the simplest and, in this sense, most "physical" seems to be the superfunction SuTra, defined in the foliowing section.

3 Asymptotic expansion and definition

In general, solution of equation (2) is not unique, even if the additional condition (3) is applied. Another holomorphic solution F of the same equation can be constructed as follows:

$$f(z) = \mathrm{SuTra}(z + s(z)) \tag{16}$$

where s is holomorphic function with period unity. The simple example is $s(z) = \varepsilon \sin(2\pi z)$ for some real constant ε. For $0 < \varepsilon < 1$, the modified superfunction still keeps the monotonous growth. The modified superfunction may have new asymptotic behaviour at infinity, because the real-holomorphic entire periodic function with real period has at least exponential growth along some lines in the direction of the imaginary axis. In order to specify the unique function SuTra, consider the construction below.

Let M be positive integer constant. Let function F have the following expansion:

$$F(z) = F_M(z) + \mathcal{O}\left(\frac{\ln(z)}{z}\right)^M \tag{17}$$

where

$$F_M(z) = -\ln(-z) + \sum_{m=1}^{M} \frac{\displaystyle\sum_{n=0}^{m} a_{m,n} \ln(-z)^n}{z^m} \tag{18}$$

and $a_{1,0}=0$. Other coefficients a are determined by substitution of equations (17),(18) into the transfer equation $\mathrm{tra}(F(z)) = F(z+1)$ and equalising the coefficients with equal powers of $\ln(-z)$ and z in the right hand side and in the left hand side of the asymptotic relation.

The coefficients a can be calculated automatically with the Mathematica software, using the code shown in Table 1. For simplicity, the evaluation of coefficients $a_{m,n}$ for $m = 1$, $m = 2$ and $m = 3$ is programmed; the extension to larger m is straightforward. Several coefficients a, calculated with this extension, are shown in Table 2.

Expression (18) can be considered as primary approximation of superfunction of the Trappmann function (1). Then, the exact solution of the transfer equation can be constructed as follows:

$$F(z) = \lim_{k\to\infty} \mathrm{tra}^k(F_M(z-k)) \tag{19}$$

In order to get superfunction, that satisfies also the additional condition (3), define

$$\mathrm{SuTra}(z) = F(z + x_0) \tag{20}$$

where $x_0 \approx -1.1259817765745026$ is real solution of equation

$$F(x_0) = 0 \tag{21}$$

As the real part of the argument gets large negative values, the derivative of the transfer function tra by (1) approaches unity. In the same limit, the contribution of the highest terms in the expansion (18) decays. Therefore, the resulting function SuTra does not depend on the number M of terms in sum in (18); in principle, value $M=1$ could be used. However, for $M=1$, the limit in (19) converges slowly. Value $M = 11$ is chosen in the next section for the numeric implementation of function SuTra.

Table 1: Mathematica code, that calculates coefficients a in expansion (17),(18)

```
T[z_] = z + Exp[z];
Clear [n, m, M];
P[m_, L_] := Sum[a[m, n] L^n, {n, 0, m}]; P[m, L];
F[z_] = -Log[-z] + a[1, 1] Log[-z]/z + Sum[ P[m, Log[-z]]/z^m, {m, 2, M}]

M = 12;
F1x = F[-1/x + 1];
Ftx = T[F[-1/x]];

s[1] = Series[F1x - Ftx, {x, 0, 2}];
t[1] = Extract[Solve [Coefficient[s[1], x^2] == 0, {a[1, 1]}], 1]
A[1, 1] = ReplaceAll[a[1, 1], t[1]];
su[1] = t[1]

m = 2; s[m] = ReplaceAll[Series[F1x - Ftx, {x, 0, m + 1}], su[m]];
t[m] = Coefficient[ReplaceAll[s[m], Log[x] -> L], x^(m + 1)];
u[m] = Collect[t[m], L];
v[m] = Table[Coefficient[u[m] L, L^(n + 1)] == 0, {n, 0, m}];
w[m] = Table[a[m, n], {n, 0, m}];
ad[m] = Extract[Solve[v[m], w[m]], 1];
su[m + 1] = Join[su[m], ad[m]];
ReplaceAll[ReplaceAll[F[x], su[m + 1]], Log[-x] -> L]

m = 3; s[m] = ReplaceAll[Series[F1x - Ftx, {x, 0, m + 1}], su[m]];
t[m] = Coefficient[ReplaceAll[s[m], Log[x] -> L], x^(m + 1)];
u[m] = Collect[t[m], L];
v[m] = Table[Coefficient[u[m] L, L^(n + 1)] == 0, {n, 0, m}];
w[m] = Table[a[m, n], {n, 0, m}];
ad[m] = Extract[Solve[v[m], w[m]], 1];
su[m + 1] = Join[su[m], ad[m]];
ReplaceAll[ReplaceAll[F[x], su[m + 1]], Log[-x] -> L]
```

Table 2: Coefficients a in the expansion (17),(18)

0	$-\frac{1}{2}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$	$a_{1,5}$	$a_{1,6}$	$a_{1,7}$	$a_{1,8}$
$\frac{1}{6}$	$-\frac{1}{4}$	$\frac{1}{8}$	$a_{2,3}$	$a_{2,4}$	$a_{2,5}$	$a_{2,6}$	$a_{2,7}$	$a_{2,8}$
$\frac{7}{48}$	$-\frac{7}{24}$	$\frac{3}{16}$	$-\frac{1}{24}$	$a_{3,4}$	$a_{3,5}$	$a_{3,6}$	$a_{3,7}$	$a_{3,8}$
$\frac{647}{4320}$	$-\frac{35}{96}$	$\frac{5}{16}$	$-\frac{11}{96}$	$\frac{1}{64}$	$a_{4,5}$	$a_{4,6}$	$a_{4,7}$	$a_{4,8}$
$\frac{1427}{8640}$	$-\frac{4163}{8640}$	$\frac{25}{48}$	$-\frac{17}{64}$	$\frac{25}{384}$	$-\frac{1}{160}$	$a_{5,6}$	$a_{5,7}$	$a_{5,8}$
$\frac{1380863}{7257600}$	$-\frac{1883}{2880}$	$\frac{5963}{6912}$	$-\frac{653}{1152}$	$\frac{305}{1536}$	$-\frac{137}{3840}$	$\frac{1}{384}$	$a_{6,7}$	$a_{6,8}$
$\frac{3278773}{14515200}$	$-\frac{2171723}{2419200}$	$\frac{97603}{69120}$	$-\frac{3961}{3456}$	$\frac{537}{1024}$	$-\frac{263}{1920}$	$\frac{49}{2560}$	$-\frac{1}{896}$	$a_{7,8}$
$\frac{251790467}{914457600}$	$-\frac{35981749}{29030400}$	$\frac{1049251}{460800}$	$-\frac{920881}{414720}$	$\frac{69953}{55296}$	$-\frac{13381}{30720}$	$\frac{4123}{46080}$	$-\frac{363}{35840}$	$\frac{1}{2048}$

4 Implementation of function SuTra

In general, any function can be declared as "special function", as soon as its properties are known and the efficient (id est, simple, robust, fast and precise) algorithm for the evaluation is supplied. One of goals of this article is to pop-up function SuTra to the set of special functions. In this section, the efficient numerical implementation for function SuTra is suggested. The code is loaded as `http://mizugadro.mydns.jp/t/index.php/Sutran.cin` and described in this section.

At large negative values of $\Re(z)$ and/or at large values of $\Im(z)$, expression $F_M(z+x_0)$ can be considered as approximation of function $\mathrm{SuTra}(z)$:

$$\mathrm{SuTra}(z) \approx F_M(z+x_0) \tag{22}$$

The precision of this approximation can be characterized with the agreement function

$$\mathcal{A}(z) = -\lg\!\left(\frac{|F_M(z{+}x_0) - \mathrm{SuTra}(z)|}{|F_M(z{+}x_0)| + |\mathrm{SuTra}(z)|}\right) \tag{23}$$

The agreement $\mathcal{A}$ indicates, how many correct decimal digits does the approximation (22) provide. For $M=11$, levels of the agreement function are shown in figure 4. Namely this value of M is used in the C++ numerical implementation `http://mizugadro.mydna.jp/t/index.php/sutran.cin`, that provides of order of 15 correct decimal digits.

The map of agreement in the figure 4 is symmetric with respect to reflection from the real axis; only the upper part of the complex plane is shown. The thick curve is built of segment of line $x=-11$, the arc of circle of radius 18, centered at point (5,0), and the half-line along $y=6$. This thick line separates the range, where the approximation qualified as "precise" (outer area) from

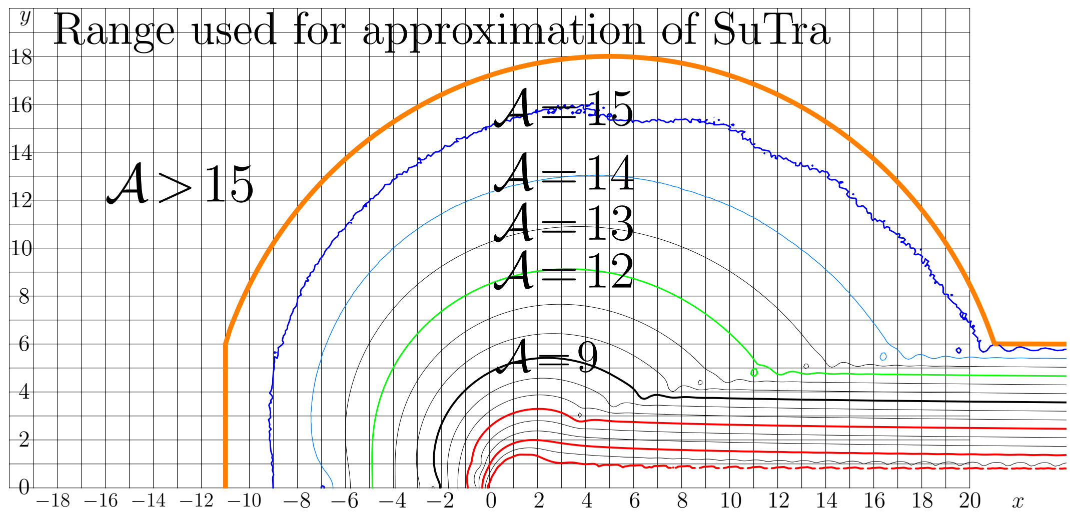

Figure 4: Map of the agreement $\mathcal{A} = \mathcal{A}(x + \mathrm{i}y)$ by (23)

the range, where the the approximation is "poor" (inner area). For values from the inner area, formula

$$\mathrm{SuTra}(z) \approx \mathrm{tra}^n(F_{11}(z+x_0-n)) \tag{24}$$

is used for a minimal natural n such that $z-n$ is at the left hand side from the thick contour. The resulting implementation returns of order of 15 significant figures.

5 Discussion

This section mention some functions, related to function SuTra defined with (18),(19),(20),(21), and the possible applications.

Once some superfunction F is defined, other superfunctions can be constructed with equation (16). They grow in the direction of the imaginary axis faster than logarithm, due to the exponential growth of the periodic function s. For these reasons, these modified functions fall out of scope of this article.

For given superfunction SuTra by (20), the inverse function, let it be called AuTra, can be constructed;

$$\mathrm{SuTra}(\mathrm{AuTra}(z)) = z \tag{25}$$

Function AuTra is Abel function for the Trappmann function tra by (1); it is described at `http://mizugadro.mydns.jp/t/index.php/AuTra` (and not included in this article, to keep it short); the expansion of AuTra can be obtained, inverting expansion (17),(18), and also with similar asymptotic analysis

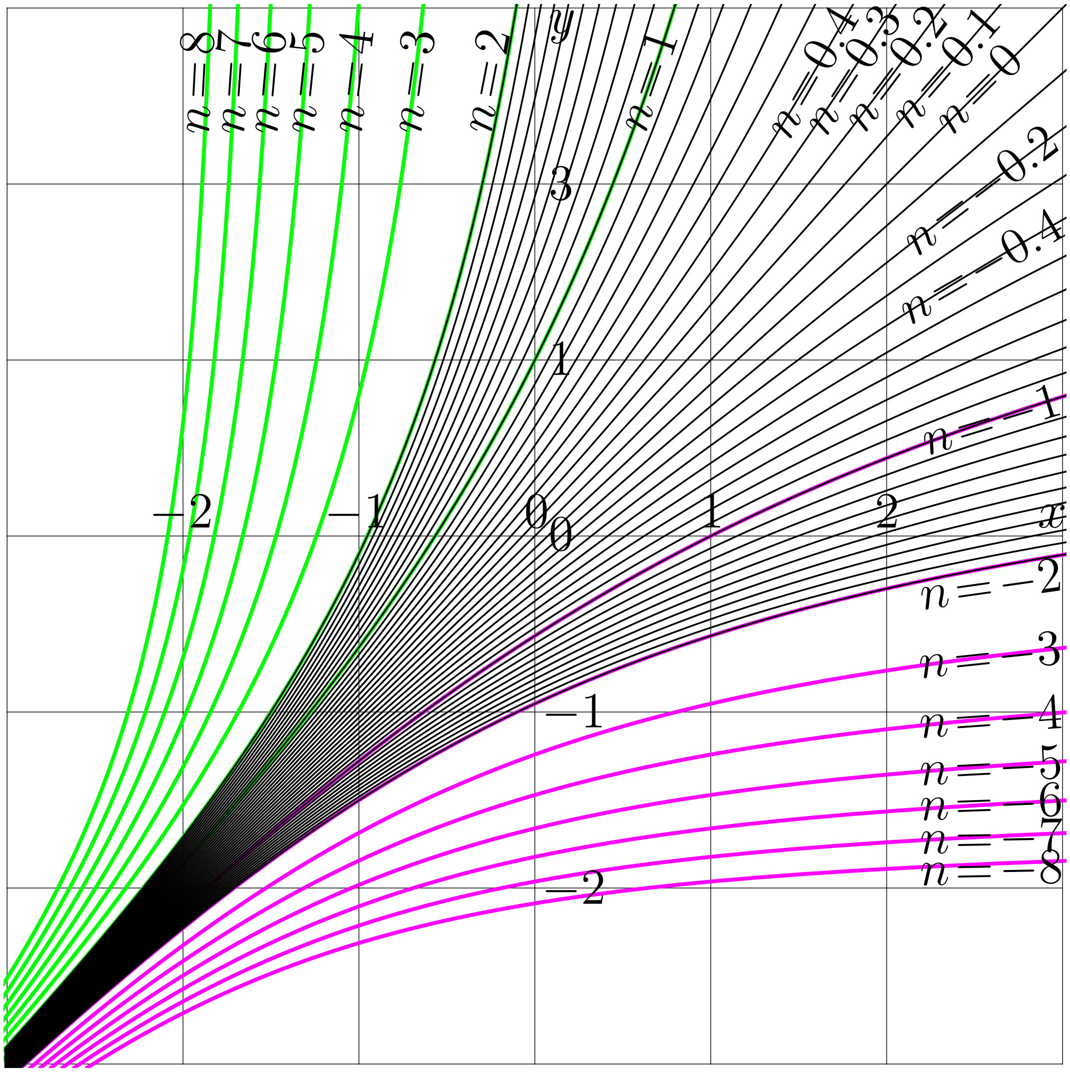

Figure 5: $y\!=\!\mathrm{tra}^n(x)$ by (27) for various n; case $n\!=\!1$ refers to $y\!=\!x\!+\!\mathrm{e}^x$.

of the Abel equation

$$\mathrm{AuTra}(\mathrm{tra}(z)) = \mathrm{AuTra}(z) + 1 \tag{26}$$

With functions SuTra and AuTra, the iterates of function tra by (1) can be expressed as follows:

$$\mathrm{tra}^n(z) = \mathrm{SuTra}(n\!+\!\mathrm{AuTra}(z)) \tag{27}$$

In this expression, the number n of iteration has no need to be integer. The iterates are shown in figure 5.

Function tra can be iterated non-integer number of times, and even complex number of times. Earlier, the similar iterates had been constructed and discussed for the exponential to various bases [10, 12, 13] and other functions [1, 14, 16] .

The Trappmann function tra by (1) is named after Henryk Trappmann, who had indicated, that it is difficult to build-up a superfunction for a transfer function without fixed points. It happened to be not really so: in wide range of values of the argument, the superfunction of tra can be expressed through superfunction of another transfer function, let it be called zex:

$$\mathrm{zex}(z) = z \exp(z) \tag{28}$$

Function zex has the real fixed point, namely, zero. For the transfer function zex, the superfunction SuZex and the Abel function AuZex = SuZex^{-1} can be constructed, using the formalism, described in [13]. Function SuZex satisfies the transfer equation

$$\mathrm{zex}(\mathrm{SuZex}(z)) = \mathrm{SuZex}(z+1) \tag{29}$$

Then in the wide range of values of z, function SuTra can be expressed as

$$\mathrm{SuTra}(z) = \ln(\mathrm{SuZex}(z)) \tag{30}$$

The inverse function AuTra = SuTra^{-1} can be expressed as

$$\mathrm{AuTra}(z) = \mathrm{AuZex}(\exp(z)) \tag{31}$$

As the SuZex had been implemented since year 2012, the first evaliations of SuTra were performed using equation (30). In particular, the representations (30),(31) through SuZex and AuZex are used to plot figure 5. However, with representation (30), the holomorphism of function SuTra in the whole complex plane is not seen. So, in this article, the representation (18), (19), (20) is suggested.

Ability to construct the non-integer iterates of a function without fixed points indicates, that the formalism of superfunctions is more powerful, than that of the Schroeder functions [27, 28] (that also allows to evaluate the same non-finiteger iterates). Various applications are expected. In particular, the charged conducing surfaces along lines $\Re(\mathrm{SuTra}(x+\mathrm{i}y) = \mathrm{const}$ could be used to emulate the logarithmic potential of a charged wire in the space without wire. The "handle", seen along the positive part of the real axis in figures 2 and 3 is unavoidable; so, the trapping of the particle should be provided in other way (see, for example, [29]). Providing the logarithmic potential without to place any material object along the centre of this potential may have application in the atom optics.

I expect, the formalism of superfunctions becomes important tool during century 21. The results above confirm the general guess, that for any physically-meaningful transfer function, the physically-meaningful superfunction can be constructed. In particular, this refers to the case, when the transfer function has no fixed points at all, as the Trappmann function (1) has no fixed points. For any growing special function, the non-integer iterate can be evaluated; the pictures, similar to figure 5 can be plotted. The non-integer iterates greatly extend the arsenal of functions, that can be used to describe the physical processes or to fit non-trivial dependences in other sciences.

6 Main Results

Function SuTra by equations (18),(19),(20),(21) and Tables 1,2 is constructed as superfunction for the transfer function $\mathrm{tra} = z \mapsto z + \mathrm{e}^z$. The analytical and numerical analysis of this function lead to the conjecture:
Function SuTra is entire and has logarithmic asymptotics, namely,
for any $\varepsilon > 0$, $\mathrm{SuTra}(z) = -\ln(-z) + \mathcal{O}(\ln(-z)/z)$ at $|z| \to \infty$
in the whole complex plane except the range $|\arg(z)| < \varepsilon$.

The complex(double) C++ implementation of function SuTra is loaded as `http://mizugadro.mydns.jp/t/index.php/Sutran.cin`. It returns 15 significant figures and allows to plot the complex maps (Figs. 2, 3) in real time.

With function SuTra, the logarithmic function can be approximated through $z \mapsto -\mathrm{SuTra}(-Sz) + \ln(S)$ for large positive values of S for all complex plane except zero and the negative part of the real axis. Up to my knowledge, before, no entire function with logarithmic asymptotics had been published, and no approximation of logarithm with entire function for huge values of the argument had been reported.

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References

[1] D.Kouznetsov, H.Trappmann. Superfunctions and square root of factorial. *Moscow University Physics Bulletin*, **65** (2010), 6 - 12.

[2] `http://www.proofwiki.org/wiki/Definition:Superfunction`

[3] `http://en.citizendium.org/wiki/Superfunction`

[4] `http://mizugadro.mydns.jp/t/index.php/Superfunction`

[5] Tools for Outstanding Research and Investigation, (2011-2013), http://mizugadro.mydns.jp/t

[6] P. Fatou. Sur les équations fonctionnelles. Bull. Math. Soc. France", **47/48** (1919 / 1920), 161 - 271 / 33 - 94, 208 - 314.

[7] P. Levi. Fonctions à croissance r'rguli'ere et it'eration f'ordre fractionnaire, Congresso Internazionale dei Matematici, Atti del Congresso Internazionale dei Matematici, Bologna 3-10 Settembre 1928, Tomo II Comunicazioni Sezione I [A-B]; Proceedings of the International Mathematical Union (1928), 277 - 282.

[8] H. Kneser. Reelle analytische Lösungen der Gleichung $\varphi(\varphi(x)) = \mathrm{e}^x$ und verwandter Funktionalgleichungen, J. Reine Angew. Math. **187** (1949), 56 - 67.

[9] W.Bergweiler. Iteration of meromorphic functions. Bulletin (New Series) of the American Mathematical society, v.29, No.2 (1993) p.151-188.

[10] D. Kouznetsov. Solution of $F(x+1) = \exp(F(x))$ in complex z-plane. Mathematics of computation **78** (2009), 1647 - 1670

[11] D.Kouznetsov. Superexponential as special function. Vladikavkaz Mathematical Journal **12** (2010), issue 2, 31 - 45. http://mizugadro.mydns.jp/PAPERS/2010vladie.pdf

[12] D.Kouznetsov, H.Trappmann. Portrait of the four regular superexponentials to base sqrt(2). Mathematics of Computation, **v.79** (2010), 1727 - 1756.

[13] H.Trappmann, D.Kouznetsov. Computation of the Two Regular Super-Exponentials to base exp(1/e). Math. Comp. **81** (2012), 2207 - 2227.

[14] D.Kouznetsov. Holomorphic extension of the logistic sequence. Moscow University Physics Bulletin **65** (2010), 91 - 98.

[15] H.Trappmann, D.Kouznetsov. Uniqueness of Analytic Abel Functions in Absence of a Real Fixed Point. Aequationes Mathematicae, **81** (2011), 65 - 76.

[16] D.Kouznetsov. Superfunctions for amplifiers. Optical Review, **20**, (2013), 321 - 326.

[17] http://en.citizendium.org/wiki/Iteration_of_function

[18] http://mizugadro.mydns.jp/t/index.php/Iteration

[19] http://mizugadro.mydns.jp/t/index.php/Abel_function

[20] http://www.proofwiki.org/wiki/Definition:Tetration

[21] http://en.citizendium.org/wiki/Tetration

[22] http://mizugadro.mydns.jp/t/index.php/Tetration

[23] http://mizugadro.mydns.jp/t/index.php/Table_of_superfunctions

[24] http://mizugadro.mydns.jp/t/index.php/Doya_function

[25] http://mizugadro.mydns.jp/t/index.php/Keller_function

[26] http://mizugadro.mydns.jp/t/index.php/Tania_function

[27] Carl C. Cowen. Iteration and the solution of functional equations for functions analytic in the unit disk. Trans. Amer. Math. Soc. **265** (1981), 69-95

[28] Carl C. Cowenoeder The Schröder equation.
http://leo.nit.ac.jp/~ishi/CMFT-Red.pdf

[29] M.Horikoshi; K.Nakagawa. Atom chip based fast production of Bose-Einstein condensate. Applied Physics B 82 (3): (2006), 363 - 366.

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