

Evaluation of holomorphic ackermanns

Dmitrii Kouznetsov

Institute for Laser Science, University of Electro-Communications, 1-5-1 Chofugaoka, Chofushi, Tokyo, 182-8585, Japan

Email address:

dima@ils.uec.ac.jp

To cite this article:

Dmitrii Kouznetsov. Evaluation of Holomorphic Ackermanns. *Applied and Computational Mathematics*. Vol. 3, No. 6, 2014, pp. 307-314. doi: 10.11648/j.acm.20140306.14

Abstract: Holomorphic extension of the Ackermann function is suggested. Algorithms of evaluation of tetration and pentation are discussed and illustrated with explicit plots and complex maps.

Keywords: Ackermann Function, Superfunction, Tetration, Pentationx

1. Introduction

The Ackermann function is known since 20th century [2]. It is often mentioned as an example of function with very fast growth [4, 18]. The Ackermann function can be defined as integer valued function A of two integer arguments, with equations

$$A(0, z) = z + 1 \tag{1}$$

$$A(m+1, 0) = A(m, 1) \text{ for } m > 0 \tag{2}$$

$$A(m+1, z+1) = A(m, A(m+1, z)) \text{ for } m > 0, z > 0 \tag{3}$$

In this article, the holomorphic extension of the Ackermann function A with respect to second argument is considered. In this consideration, the key point is the formulation of set of additional requirements, which should be added to equation (3), in order to make solution $A(m, z)$ unique for the set of complex z , while m is still assumed to be natural number.

For $m = 1$, $m = 2$ and $m = 3$, values of $A(m, z)$ can be expressed through addition, multiplication and exponentiation. In such a way, for the three values of the first argument, the generalization to the complex values of the last argument is trivial. Below, the generalization of $A(m, z)$ is suggested for complex z at $m = 4$ and $m = 5$. In such a way, the first argument of the Ackermann function still remains positive integer.

2. Basic Equations

In order to extend (generalize) the Ackermann function, I need a little bit different notations. These notations are

defined in this section.

Consider set of holomorphic functions $A_{b,m}$ for real $b > 1$ and integer $m > 0$. In this article, I call these functions ackermanns, with lowercase letters, in order to avoid confusion with the conventional, canonical Ackermann function by equations (1),(2),(3). Parameter b is called base, and integer m is considered as number of the ackermann A_b , assuming the sequence of functions $\{A_{b,1}, A_{b,2}, A_{b,3}, A_{b,4}, A_{b,5}, \dots\}$. Let these functions, id est, ackermanns A_b , satisfy equations

$$A_{b,1}(z) = b + z \tag{4}$$

$$A_{b,m}(1) = b \text{ for } m > 1 \tag{5}$$

$$A_{b,m}(z+1) = A_{b,m-1}(A_{b,m}(z)) \text{ for } m > 2 \tag{6}$$

for some ranges of complex values of z , and let these functions be holomorphic in these ranges.

For $b = 2$, it is possible to establish the relation between the classical Ackermann function A by (1),(2),(3) and ackermanns $A_{b,m}$ by (4), (5), (6):

$$A(m, z) = A_{2,m}(z+3) - 3 \tag{7}$$

In this sense, the ackermanns A are generalisations of the conventional Ackermann function A . I use different fonts for A and A in order to simplify the identification.

The comparison of the Ackermann function A to the ackermanns A is shown in Figure 1. There, $y = A(m, x)$ and $y = A_{2,m}(x)$ are plotted versus x for $m = 1, m = 2, m = 3, m = 4$. For the last value, the ackermann is interpreted as binary tetration; it is evaluated in analogy with the natural tetration [5, 20, 21].

In general, solutions A_b of equations (4),(5),(6) are not

unique. From the intuitive wish to express the solution in terms of simple elementary functions, one can suggest expressions for some of ackermanns. The First one, id est, $A_{b,1}$ is determined by (4); it is just addition of a constant, equal to base b . Then, the Second and Third ackermanns can be specified as follows:

$$A_{b,2}(x) = b z \tag{8}$$

$$A_{b,3}(x) = b^z \tag{9}$$

Parameter b is called “base”, in analogy with base of the exponential and that of logarithm. In such a way, the Second ackermann is multiplication to base b , and the Third ackermann is exponentiation to base b .

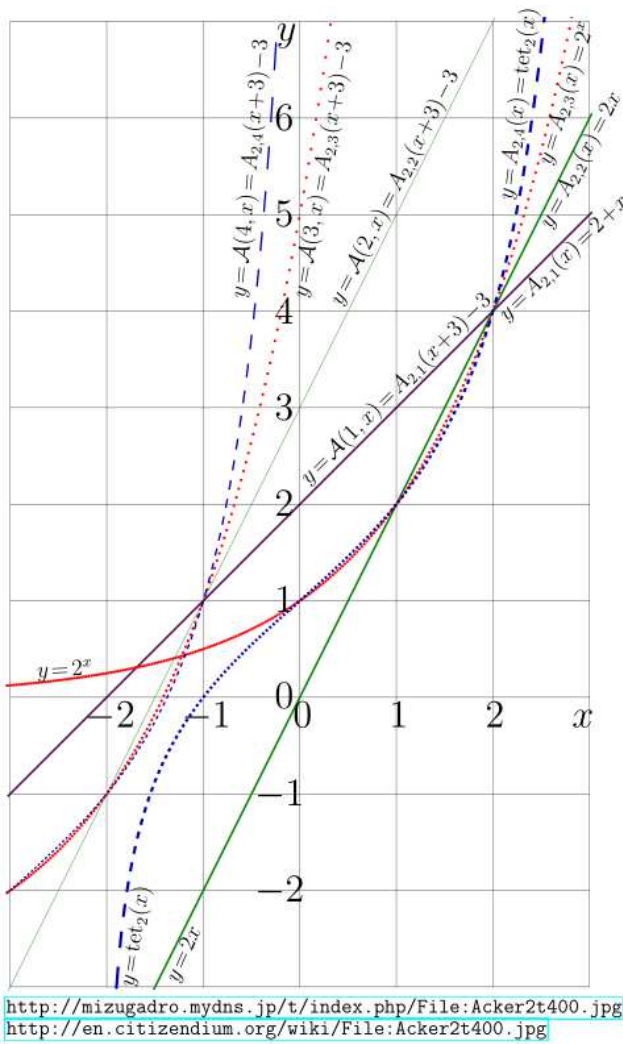


Figure 1. Comparison of notations for binary ackermanns

Equation (6) can be interpreted as a transfer equation [14, 22]. Each ackermann (except First) is superfunction [9, 18, 19] with respect to the previous one; and each ackermann can be considered as a transfer function for the next one. Certain results about construction and evaluation of superfunctions had been reported in the beginning of 21th century [5-15].

Several methods of construction and evaluation of superfunctions have been suggested and implemented. Here, these methods are used to evaluate the Fourth and Fifth ackermanns, that cannot be defined with elementary functions (as the Second and Third ackermanns by eq. (8) and (9)).

There is an important question about choice of the additional conditions, additional requirements, that should be added to equations (4), (5), (6) in order to provide the uniqueness of the solution. These conditions should allow to evaluate the functions and to plot the figures. I suggest hints for these additional conditions and I illustrate these with evaluation of the Fourth ackermann, which already has special name, tetration:

$$A_{b,4} = \text{tet}_b \tag{10}$$

Then, for base $b = e = \exp(1) \approx 2.71$, I construct the 5th ackermann, let it be called pentation:

$$A_{b,5} = \text{pen}_b \tag{11}$$

For the constructed functions, I provide the efficient algorithms of evaluation, real-real plots and the complex maps. These plots and maps are loaded to sites TORI and Citizendium and supplied with their generators. (Colleagues can load the generators and reproduce figures from this article, as well as make new illustrations and/or use these codes for other applications.) Then I formulate suggestions for the future work on development of ackermanns and other superfunctions. The superfunctions should greatly extend the arsenal of holomorphic functions available for description of physical phenomena and approximation of functions; for this reason I consider the topic as interesting and important.

For base $b = 2$, the ackermanns $A_{b,m}$ by (4), (5), (6) and the canonical Ackermann function A by (1), (2), (3) are related through equation (7); this relation is illustrated in Figure 1. Each new ackermann A_2 , constructed here, appears as generalization of the canonical Ackermann function, mentioned in the Abstract.

3. Tetration

Tetration $f = A_{b,4} = \text{tet}_b$ is the 4th ackermann. It is superfunction of exponentiation; it satisfies the transfer equation with the condition at zero below:

$$f(z+1) = b^{f(z)} \tag{12}$$

$$f(0) = 1 \tag{13}$$

Complex maps of tetration are shown in Figures 2, 3 for various values of base b . The evaluation is described in [5, 6, 7, 8, 11, 12].

Maps for $b = 2 \approx 1.41$, $b = \exp(1/e) \approx 1.44$ and $b = 1.5$ are shown in figure 2. Similar maps for $b = 2$, $b = e \approx 2.71$, and for $b = 1.52598338517 + 0.0178411853321 i$ are shown in Figure 3.

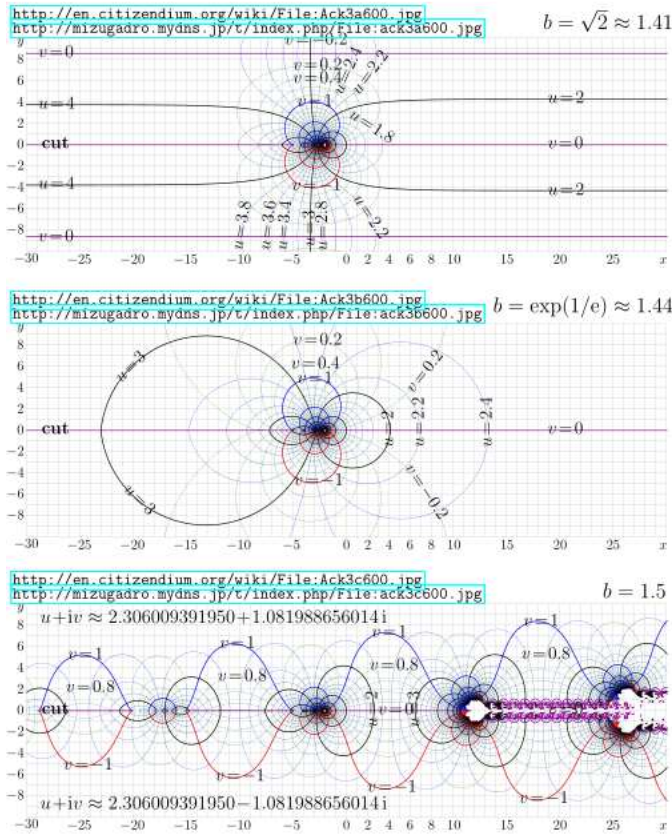


Figure 2. $u+iv = \text{tet}_b(x+iy)$ for $b=2$, $b = \exp(1/e)$, and $b=1.5$

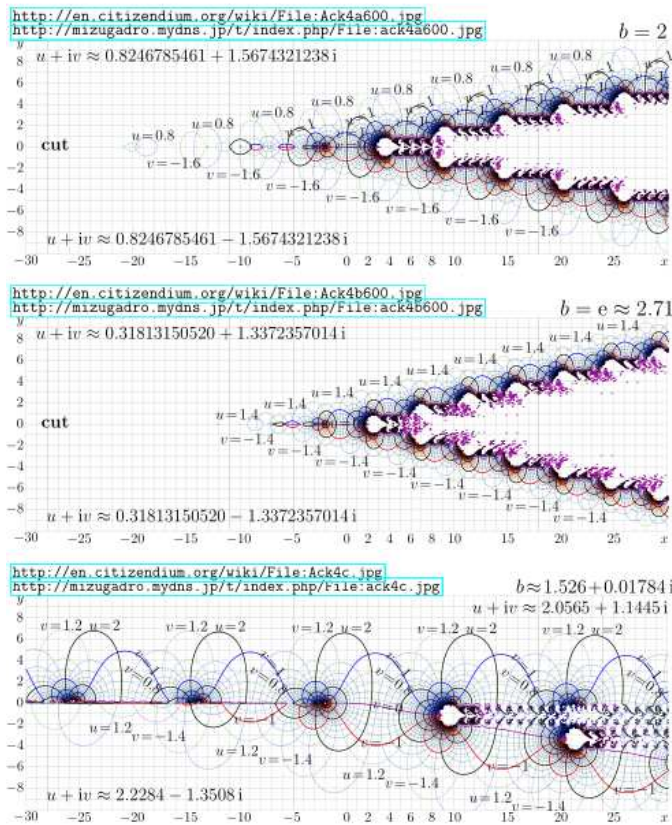


Figure 3. $u+iv = \text{tet}_b(x+iy)$ for $b=2$, for $b = e \approx 2.71$, and for $b = 1.52598338517 + 0.0178411853321 i$

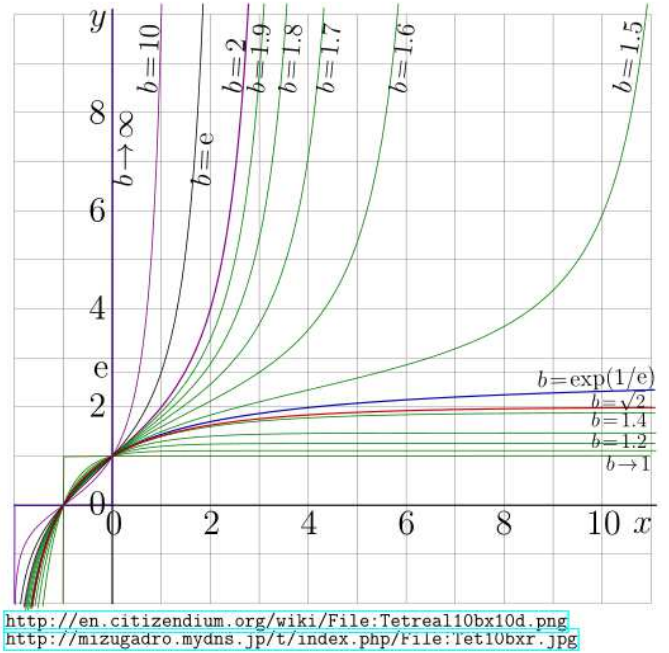


Figure 4. Graphics of tetration: $y = \text{tet}_b(x)$ versus x for various b

The algorithm of the calculation of tet_b for $1 < b < \exp(1/e)$ is described in the articles [8, 9, 10]. The algorithm for $b = \exp(1/e)$ is described in [12]. That for $b > \exp(1/e)$ is described in [5, 6].

With minimal modification, this can be applied for complex values of base b . This adaptation is straightforward; the example is shown at the bottom map of Figure 3.

The interpretation is most simple for real base $b > 1$. In order to provide the uniqueness of the solution, it is assumed that the tetration $\text{tet}_b(z)$ is holomorphic at least in the range $\text{Re}(z) > -2$, and it is bounded at least in the range $|\text{Im}(z)| \leq 1$. Conjecture about existence and uniqueness of such tetration (and uniqueness of its inverse function $\text{ate}_b = \text{tet}_b^{-1}$) had been considered for various values of base b , in attempts to find any contradiction (and to negate the conjecture) [5, 6, 7, 8, 11, 12]. Some theorems about the uniqueness are suggested in the recent publication [11].

Figures 2 and 3 show how does the complex map of tetration tet_b modify, as the base b gradually changes from small values (of order of unity) to larger values. The behavior for real values of argument is shown in Figure 4 and discussed in the next section.

4. Tetration for Real Base and Real Argument

This section deals with the 4th ackermann, $A_{b,4} = \text{tet}_b$, for $b > 1$. Explicit plots $y = \text{tet}_b(x)$ versus x are shown in Figure 4 for various values of base b .

For moderate values of the argument, tetration can be approximated with elementary function. This approximation is described below. Let

$$\begin{aligned}
 d &= \ln(b) \\
 q &= \sqrt{d} \\
 c_0 &= -1.0018 + \frac{0.1512848482(1.+33.0471529885q-3.51771875598d)q}{1+3.2255053261256337q} + \frac{\ln(2)-\frac{1}{2}}{d} \\
 c_1 &= 1.1 - 2.608785958462561(1 - 0.6663562294911147q)q - \frac{\ln(2)-\frac{5}{8}}{d} \\
 c_2 &= -0.96 + 3.0912038297987596 \frac{(1+0.60213980487853d)q}{1+4.24046755648d} + \frac{\ln(2)-\frac{3}{4}}{d} \\
 c_3 &= 1.2 - 10.44604984418533 \frac{(1+0.213756892843q+0.369327525447d)q}{1+4.9571563666q+7.702332166d} - \frac{\ln(2)-\frac{131}{192}}{d} \\
 \text{fit}_b(z) &= (1+c_0z+c_1z^2+c_2z^3+c_3z^4)(z+1) + \ln(z+2) - \frac{\ln(2)}{d}(1+z)
 \end{aligned}
 \tag{14}$$

Function fit by (14) is constructed, approximating tetration, evaluated with algorithms described previously [5, 6, 8, 9, 15]. At $1.05 < b < 5$, function $\text{fit}_b(z)$ approximates function $\text{tet}_b(z)$ for $|z| < 1$ with 4 significant figures. Taking into account the transfer equation (12), this approximation can be extended to the strip along the real axis. In particular, the resulting approximation is valid along the real axis. This representation is used to plot the most of curves in Figure 4, the error of approximation is not seen even at the zooming-in of the plot; but for $b = 10$ the original representation of tetration through the Cauchi integral [5] is involved.

Figure 4 shows behavior of the 4th ackermann of real argument to real base $b > 1$. This function has logarithmic singularity at -2 . The graphic passes through points $(-1, 0)$, $(0, 1)$, $(1, b)$. At large positive values of argument, the 4th ackermann quickly grows up for $b > \exp(1/e)$, and approaches the largest fixed point of exponent for $b \leq \exp(1/e)$.

The approximation (14) used to plot Figure 4 can be a prototype of the internal implementation of tetration to real base $b > 1$ in the programming languages. The algorithm for the precise evaluation [5, 12, 8] is exact in the sense that it allows to evaluate tetration with any required precision. In such a way, both, the range of validity and the precision could be improved. One of direction of the future work can be covering of the whole complex plane, both in base b and argument z , with the precise and efficient approximations of tetration $\text{tet}_b(z)$; the expansions (including the truncated Taylor series) should provide better performance than the original algorithm through the Cauchi integral [5] for complex base b and real $b > \exp(1/e)$.

Figures 2 and 3 show how does the complex map of tetration tet_b changes as the base b gradually changes from small values (of order of unity) to large values. For same real values of base b , the corresponding real-real plots are shown in Figure 4. The efficient algorithms of evaluation of tetration are supplied, so, the tetration should be interpreted as elementary function.

5. Natural Tetration and Pentation

As an example of generalisation of ackermanns, in this section I describe the natural pentation and compare it to the natural tetration [5]. The 4th ackermann is already described and implemented for various bases [5, 6, 12, 8], and the same

methods can be used to built-up the pentation, $\text{pen}_b = A_{b,5}$.

Here, I consider the only natural pentation, id est, that to base $b = e$. It is superfunction of tetration; complex map of tetration is repeated in Figure 6. It is zoom-in of the map of natural tetration at Figure 2; it can be compared to the map of pentation at Figure 7. In order to provide uniqueness of a superfunction, one should indicate the way of the construction that specifies its asymptotic behaviour at large values of argument. Let the superfunction F of tetration approaches the fixed point

$$L \approx -1.85035452902718 \tag{15}$$

of tetration, id est, the real solution of equation

$$\text{tet}(L) = L \tag{16}$$

at infinity. The graphical solution of equation (16) is shown in Figure 5.

Figure 5 can be considered as simplified zoom-in from the central part of Figure 4, with additional line $y = x$. As tetration can be evaluated with arbitrary precision, solution L of equation (16) also can be found with arbitrary precision; in this sense L is exact quantity. In order to see the details, the zoom-in of the complex map of the natural tetration is repeated in Figure 6, in the same notations, as in Figures 2 and 3. The complex map of thetetration in Figure 6 can be compared to the map of pentation, shown in Figure 7 in the same notation. Construction and evaluation of pentation is described below.

In Figure 5, the fixed point of tetration is denoted as $L_{e,4,0} = L$, in order to distinguish it from the fixed points of other ackermanns. The subscripts indicates that this quantity correspond to the natural ackermann, id est, to the ackermann to base $b = e$; that it refers to the Foruth ackermann, id est, tetration, and that it refers to the minimal real solution. Other supertetrations can be constructed using other fixed points, that are seen in the Figure 5, but not marked; they could be denoted $L_{e,4,1}$ and $L_{e,4,2}$.

Only one among varies superfunctions of tetration is called ‘‘pentation’’. I construct this pentation in the following way. I search for the solution F of the transfer equation

$$F(z+1) = \text{tet}(F(z)) \tag{17}$$

as the following expansion:

$$F(z) = f(z) + O(\varepsilon^M) \tag{18}$$

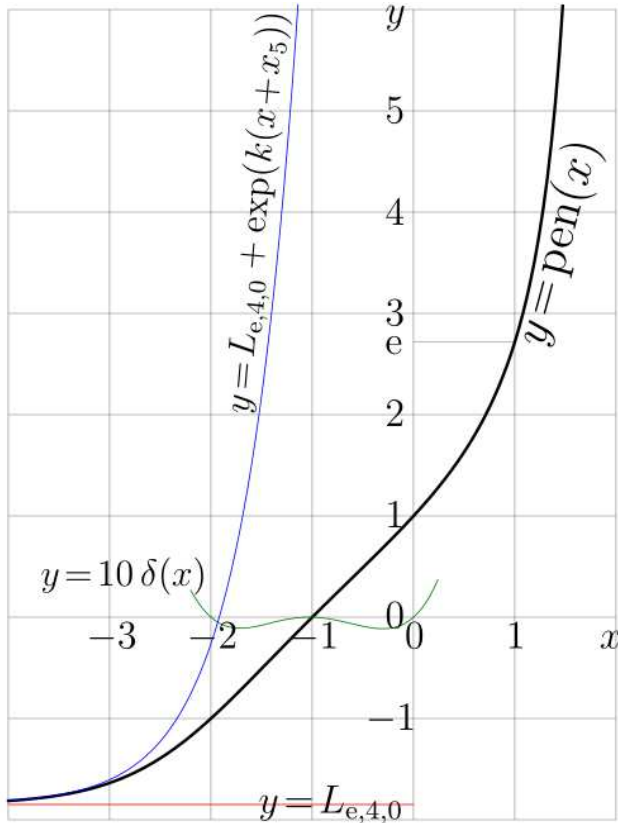
where

$$f(z) = L + \sum_{m=1}^{M-1} a_m \varepsilon^m \tag{19}$$

and

$$\varepsilon = \exp(kz) \tag{20}$$

1. Complex map of this pentation is shown in Figure 7, announced above.



<http://mizugadro.mydns.jp/t/index.php/File:Penplot.jpg>
<http://en.citizendium.org/wiki/File:Penplot.jpg>

Figure 8. $y = \text{pen}(x)$ by (26), its asymptotic by (27) and error of the linear

The real-real plot of pentation by (26) is shown in Figure 8 with thick curve. The additional horizontal line $y = L_{e,4,0}$ shows the asymptotic at large negative values of the argument. More advanced asymptotic, id est, the first approximation

$$y = L_{e,4,0} + \exp(k(x+x_5)) \tag{27}$$

is also shown with thin curve. The curve for pentation lies between these two asymptotics.

In vicinity of the segment $-2 < x < 0$, pentation $y = \text{pen}(x)$ can be approximated with linear function $y = x + 1$. Error of this approximation can be characterized with deviation

$$\delta(x) = \text{pen}(x) - (x+1) \tag{28}$$

In order to show this deviation in Figure 8, it is scaled up with factor 10. The same linear approximation had been suggested [3] also for the previous ackermann, id est, for tetration $\text{tet} = A_{e,4}$. This approximation provides at least two significant figures in the range specified, and the curve of titration looks almost straight in the interval specified. The complex map of pentation in Figure 7 can be compared to the map of tetration in Figure 6.

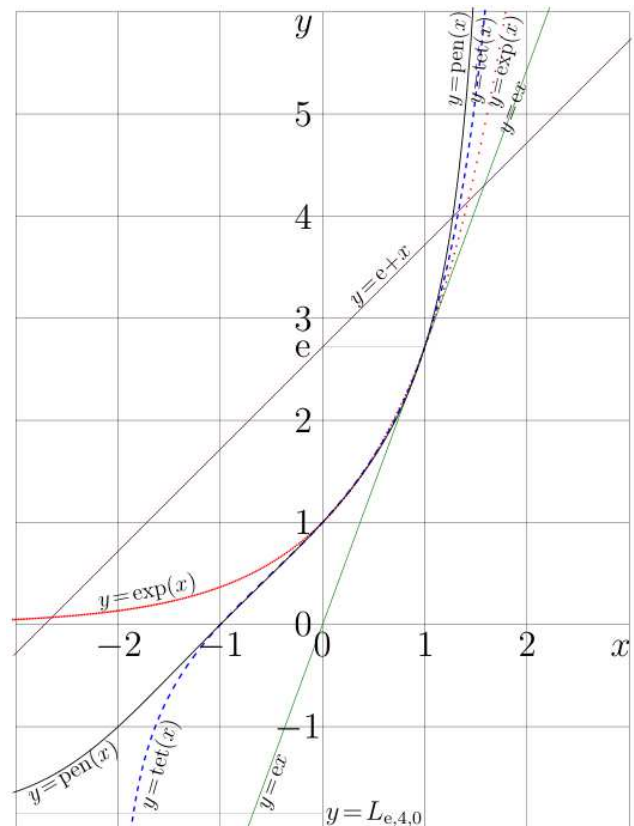
The complex map of natural tetration in figure 6 had been

published previously [5, 6]. As for the map of pentation in figure 7, it is new; up to my knowledge, no complex map of pentation had been published at least before year 2014 [15].

Tetration $\text{tet}(z)$ is holomorphic in the whole complex plane except the half-line $z \leq -2$. Pentation is holomorphic at least for $|\text{Im}(z)| < |P|/2 \approx 1.683838$, where $P = 2\pi i/k$ is period; pentation, as exponential, is periodic function. A little bit more than two periods are covered by the range of the map at Figure 7. Pentation has the countable set of cut lines, parallel to the real axis. In Figure 7, these cuts are marked with dashed lines.

The first 5 natural ackermanns are compared in Figure 9 for real values of the argument. It corresponds to base $b = e$. The curve for pentation pen is borrowed from Figure 8. Similar figures can be plotted also for other values of base b .

In the Mathematica software, there is already appropriate name for the procedure to calculate iterates of a function; it could be called ‘Nest’. In the current version, this procedure does not include construction of superfunction nor its inverse (abelfunction [1, 11, 16]), and it deals only with cases, when the number of iterates is expressed with integer constant. The generalization should allow to call the Nest with any argument and even to perform the numerical evaluation, if the number of iterates is expressed with a complex constant. Then, the automatic construction and evaluation of the highest ackermanns should be good test to verify and debug such a procedure, and this may be matter for the future work.



<http://mizugadro.mydns.jp/t/index.php/File:Ackerplot400.jpg>
<http://en.citizendium.org/wiki/File:Ackerplot400.jpg>

Figure 9. $y = A_{e,m}(x)$ versus x for $m = 1, 2, 3, 4, 5$

6. Conclusion

Generalization of the Ackermann function can be performed through the formalism of superfunctions [9, 18, 19]. Notation “ackermann” is suggested for this generalization. Sequence of the ackermanns is suggested. Each of new ackermanns (except First) appears as superfunction of the previous one, and as transfer function for the next one; this relation is expressed with the transfer equation (6). The already reported methods [5, 12, 8, 14] of construction of superfunctions can be used to build-up the chain of the holomorphic ackermanns. The classical binary Ackermann A function appears as the special case, it is expressed through the ackermanns $A_{2,m}$ with equation (7). This relation is illustrated in Figure 1.

The first 5 ackermanns already have special names: addition, multiplication, exponentiation, tetration and pentation. For base $b = 2$, four of these functions are plotted in Figure 1. For natural base $b = e$, these functions are plotted in Figure 9, and the complex maps of tetration tet and pentation pen are shown in Figures 6 and 7.

In order to provide uniqueness of holomorphic ackermanns, their asymptotic behavior should be specified. The natural choice is requirement that each new ackermann (except first and second) exponentially approaches the fixed point of the previous one; for pentation, it is determined by asymptotic (18), (19). The choice of the fixed point determines the superfunction.

The future work may include elaboration of algorithms for the automatic construction of superfunctions. Building up the chain of holomorphic ackermanns as solutions of equations (4), (5), (6) may be a good example for testing and debugging of such a procedure.

References

- [1] http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=PPN243919689_0001
N.H.Abel. Untersuchung der Functionen zweier unabhängig veränderlicher Größen x und y , wie $f(x,y)$, welche die Eigenschaft haben, dafs $f(z,f(x,y))$ eine symmetrische Function von z , x und y ist. Journal für die reine und angewandte Mathematik, V.1 (1826) Z.1115.
- [2] <http://link.springer.com/article/10.1007%2FBF01459088>
Wilhelm Ackermann. Zum Hilbertschen Aufbau der reellen Zahlen. Mathematische Annalen 99, Number 1(1928), Z.118-133.
- [3] <http://www.tandfonline.com/doi/abs/10.1080/106524605000422247#.V17FoBaD5N2> M.H.Hooshmand, August 2006, "Ultra power and ultra exponential functions", Integral Transforms and Special Functions, Vol. 17, No. 8, 549-558.
- [4] <http://eusebeia.dyndns.org/veblen/veblen> On Fast-growing Computable Functions. 07 May 2008.
- [5] <http://www.ams.org/journals/mcom/2009-78-267/S0025-5718-09-02188-7/home.html>
<http://www.ils.uec.ac.jp/~dima/PAPERS/2009analuxpRepri.pdf>
f <http://mizugadro.mydns.jp/PAPERS/2009analuxpRepri.pdf>
D.Kouznetsov. Analytic solution of $F(z+1)=\exp(F(z))$ in complex z -plane. Mathematics of Computation 78 (2009), 1647-1670.
- [6] <http://mizugadro.mydns.jp/PAPERS/2009vladie.pdf>
<http://www.ils.uec.ac.jp/~dima/PAPERS/2009vladie.pdf>
D.Kouznetsov. Tetration as special function. Vladikavkazskii Matematicheskii Zhurnal, 2010, v.12. issue 2, p.31-45.
- [7] <https://bbuseruploads.s3.amazonaws.com/bo198214/bunch/downloads/main.pdf> Henryk Trappmann, Dmitrii Kouznetsov. 5+ methods for real analytic tetration. June 28, 2010.
- [8] <http://www.ams.org/journals/mcom/2010-79-271/S0025-5718-10-02342-2/home.html>
https://www.researchgate.net/publication/220576905_Portrait_of_the_four_regular_super-exponentials_to_base_sqrt%282%29
http://eretrandre.org/rb/files/Kouznetsov2009_215.pdf
<http://www.ils.uec.ac.jp/~dima/PAPERS/2010q2.pdf>
<http://mizugadro.mydns.jp/PAPERS/2010q2.pdf>
D.Kouznetsov, H.Trappmann. Portrait of the four regular superexponentials to base $\sqrt{2}$. Mathematics of Computation, 2010, v.79, p.1727-1756.
- [9] <http://link.springer.com/article/10.3103%2FS0027134910010029>
https://www.researchgate.net/publication/226675861_Superfunctions_and_sqrt_of_factorial D.Kouznetsov, H.Trappmann. Superfunctions and square root of factorial. Moscow University Physics Bulletin, 2010, v.65, No.1, p.6-12. (Russian version: p.8-14)
- [10] <http://link.springer.com/article/10.3103%2FS0027134910020049#page-2>
<http://www.ils.uec.ac.jp/~dima/PAPERS/2010logistie.pdf>
<http://mizugadro.mydns.jp/PAPERS/2010logistie.pdf>
D.Kouznetsov. Holomorphic extension of the logistic sequece. Moscow University Physics Bulletin, 2010, series 3, issue 2, page 23-30
- [11] <http://link.springer.com/article/10.1007%2Fs00010-010-0021-6>
<http://www.ils.uec.ac.jp/~dima/PAPERS/2011uniabel.pdf>
https://www.researchgate.net/publication/226947838_Holomorphic_extension_of_the_logistic_sequence
<http://mizugadro.mydns.jp/PAPERS/2011uniabel.pdf>
H.Trappmann, D.Kouznetsov. Uniqueness of Analytic Abel Functions in Absence of a Real Fixed Point. Aequationes Mathematicae, v.81, p.65-76 (2011)
- [12] <http://www.ams.org/journals/mcom/0000-000-00/S0025-5718-2012-02590-7/S0025-5718-2012-02590-7.pdf>
https://www.researchgate.net/publication/51892955_Computation_of_the_Two_Regular_Super-Exponentials_to_base_exp%281e%29
<http://www.ils.uec.ac.jp/~dima/PAPERS/2012e1eMcom2590.pdf>
<http://mizugadro.mydns.jp/PAPERS/2012e1eMcom2590.pdf>
H.Trappmann, D.Kouznetsov. Computation of the Two Regular Super-Exponentials to base $\exp(1/e)$. Mathematics of Computation, 2012, 81, p.2207-2227.
- [13] <http://link.springer.com/article/10.1007/s10043-013-0058-6>
<http://www.ils.uec.ac.jp/~dima/PAPERS/2012or.pdf>
<http://mizugadro.mydns.jp/PAPERS/2012or.pdf> Dmitrii Kouznetsov. Superfunctions for optical amplifiers. Optical Review, July 2013, Volume 20, Issue 4, pp 321-326.

- [14] <http://www.m-hikari.com/ams/ams-2013/ams-129-132-2013/kouznetsovAMS129-132-2013.pdf>
<http://www.ils.uec.ac.jp/~dima/PAPERS/2013hikari.pdf>
<http://mizugadro.mydns.jp/PAPERS/2013hikari.pdf>
D.Kouznetsov. Entire function with logarithmic asymptotic. Applied Mathematical Sciences, 2013, v.7, No.131, p.6527-6541.
- [15] http://mizugadro.mydns.jp/t/index.php?title=Superfunctions_in_Russian <http://www.ils.uec.ac.jp/~dima/BOOK/202.pdf>
<http://mizugadro.mydns.jp/BOOK/202.pdf> D.Kouznetsov. Superfunctions. LAP, 2014.08.01. ISBN 3659562025, in Russian.
- [16] <http://mizugadro.mydns.jp/t/index.php/Abelfunction>
- [17] <http://mathworld.wolfram.com/AckermannFunction.html>
- [18] <http://mizugadro.mydns.jp/t/index.php/Superfunction>
- [19] <http://en.citizendium.org/wiki/Superfunction>
- [20] <http://mizugadro.mydns.jp/t/index.php/Tetration>
- [21] <http://en.citizendium.org/wiki/Tetrational>
- [22] http://mizugadro.mydns.jp/t/index.php/Transfer_equation
- [23] <http://www.ils.uec.ac.jp/~dima/2014ACKER/> Latex version and the C++ generators of figures of this article.