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SUPER SIN

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Abstract

Iterates of function sin are considered. The superfunction SuSin is constructed as holomorphic solution of the transfer equation $\sin(\mathrm{SuSin}(z)) = \mathrm{SuSin}(z+1)$. The Abel function AuSin is constructed as solution of the Abel equation $\mathrm{AuSin}(\sin(z)) = \mathrm{AuSin}(z) + 1$; in wide range of values z, the relation $\mathrm{SuSin}(\mathrm{AuSin}(z)) = z$ holds. Iteration of sin is expressed with $\sin^n(z) = \mathrm{SuSin}(n + \mathrm{AuSin}(z))$, where the number n of iteration has no need to be integer. The efficient algorithms for evaluation of functions SuSin and AuSin are suggested and implemented as complex double routines in C++. Complex maps of these functions are supplied.

1. Introduction

Needs and interest to consider the non-integer iterates of holomorphic functions may be related to description of the chaotic dynamics of systems, that are characterized with a transfer function. In the simplest case, the transfer function is just a single-value holomorphic function of a single variable. Then the iterates of the transfer function may correspond to the

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observation of the system at the discrete moments of time. In this case, the system may show stochastic behaviour. This behaviour becomes regular as soon as we interpret time, which has sense of the number of iterations, as continuous parameter. This motivation had been mentioned many times [5, 16, 7, 12, 11]; attempts to evaluate the non-integer iterates are traced through centuries, since works by Abel [1] and Kneser [3].

In century 20, many algorithms for evaluation of non-integer iterates had been suggested. Some references are collected recently by Trappmann and Kouznetsov [15], and earlier, even more references had been suggested by Bergweiler [6]. The common feature of early algorithms for the non-integer iterates is, that they are almost unusable: at the numerical implementation, the CPU time is large, and the precision of evaluation is poor. No complex map to any non-trivial iterated function had been presented until years 2009, 2010; when the efficient algorithms for the evaluation of superfunctions and the Abel functions had been implemented and the half iterates of the exponential and factorial were calculated, plotted and published [8-11, 15]. However, from the point of view of application in Physics (see, for example, [19]), the robust and efficient (fast and precise) algorithms are quire desirable, in order to deal with the superfunctions, the Abel functions and the non-integer iterate in a way similar to that one treats other special functions. In such a way, the mathematical formalism should be developed and adjusted.

Superfunctions and Abel functions are declared as tools of the scientific research in the 2009 at the Moscow University Physics Bulletin [11]. It is expected, that for any physically-meaningful transfer function, the reasonable, physically-meaningful superfunction can be constructed. In many cases, the non-integer iterates can be constructed also through the Schröder functions [13, 18, 17], and iterates of sin can be treated with the Schröder equation too. However, the formalism of superfunctions is more robust, precise (only the rough approximations are suggested at sites [18, 17]) and more universal. In particular, with superfunctions, we can construct iterates for the transfer functions without real fixed points [8, 9, 14], and even for the transfer function without any fixed point [21]. For this reason, in this article, the iterates are constructed with superfunction and the Abel function, and the Schröder functions are not considered.

The superfunctions are expected to have many applications; they may become so important, as the exponentiation became one of the most important tools for the scientific analysis since century 19. The simple application for the laser science is already suggested [19], many others are discussed [11].

For the applications, first, the behaviour of functions for the real values of the argument is important. Nevertheless, the functions should be constructed also for the complex values. Behaviour at the complex plane is essential to discriminate the solution and choose the simplest one. Some algorithms explicitly use this behaviour for the evaluation [8]. In addition, the algorithms can be boosted with the Taylor series [9], and the Cauchi integral formula allows the efficient evaluation of the coefficients in the expansions; again, evaluation of function in the complex plane is important. For these reasons, as in previous publications [8-12, 14, 21], here the functions are considered for the complex arguments.

2. Basic Formulas

For some holomorphic function T, referred here as the transfer function, define the superfunction F as holomorphic solution of the transfer equation

$$T(F(z)) = F(z+1) \tag{1}$$

and let the Abel function G be inverse of the superfunction, $G = F^{-1}$. Here, the number in superscript after the name of function indicates the number of iteration, and never indicates the power function. In these notations, $P^2(\psi) = P(P(\psi))$, but never $P(\psi)^2$; and $\sin^2(x) = \sin(\sin(x))$, but never $\sin(x)^2$. This notation is used since century 20 [6]. One can easily check that the Abel function satisfies the Abel equation,

$$G(T(z)) = G(z) + 1. \tag{2}$$

Once the superfunction F and the Abel function G are established, the nth iteration of the transfer function T can be defined with

$$T^n(z) = F(n + G(z)) \tag{3}$$

in such a way, that for certain range of values of z, the relation $T^m(T^n(z)) = T^{m+n}(z)$ holds. As the iterate is defined with equation (3), number n of iteration has no need to be integer.

For a given transfer function, the superfunction is not unique. The different superfunctions may give different iterates. Consideration of the iterates in the complex plane is important to choose the superfunction, different from the constant, that has simplest behaviour at infinity. In this article, the superfunction of sin, or super sin, is constructed and denoted with name SuSin; it is assumed, that for large values of z,

$$\mathrm{SuSin}(z) = \sqrt{\frac{3}{z}}\left(1 + O\left(\frac{\ln(z)}{z}\right)\right). \tag{4}$$

After to force the search engine, I found that the leading factor $\sqrt{\frac{3}{z}}$ had been already suggested; Hemsidor [17] uses name "Niclas Carlsson formula" for expression $\mathrm{SuSin}(z) \approx \sqrt{\frac{3}{z}}$.

In this article, the superfunction SuSin, satisfying relation (4), is constructed, and the algorithm for the evaluation is suggested, and many terms are added instead of O in (4). In such a way, this article can be considered as verification (in the sense of the TORI axioms [20]) of the Niclas Carlsson formula [17].

3. Super Sin

Construction of superfunction for the transfer function $T = \sin$ is similar to that for the transfer functions $T(z) = \exp\left(\frac{1}{\mathrm{e}} z\right)$ and $T(z) = \mathrm{tra}(z) = z + \exp(z)$, described recently [15, 21]. Search for the solution f of the transfer equation

$$\sin(F(z)) = F(z+1) \tag{5}$$

in the following form:

$$f(z) = F_M(z) + O\left(\frac{\ln(z)^{M+1}}{z^{M+3/2}}\right), \tag{6}$$

where

$$F_M(z) = \sqrt{\frac{3}{z}}\left(1 - \frac{3\ln(z)}{10z} + \sum_{m=2}^{M} P_m(\ln(z))\, z^{-m}\right), \tag{7}$$

$$P_m(z) = \sum_{n=0}^{m} a_{m,n} z^m \tag{8}$$

and coefficients a are constants.

One can guess the factor $\sqrt{3/z}$ above even without to read the references mentioned: expand the transfer function at zero, replace $F(z+1)$ to $F(z) + F'(z)$ and solve the resulting differential equation; the solution indicates the leading term of the expansion of the superfunction. In the similar way, one can guess value of the coefficient $-3/10$ at the first term. In order to calculate other coefficients a, the asymptotic expansion should be substituted into the transfer equation. This can be done automatically with the Mathematica code below:

```
P[m_, L_] := Sum[a[m, n] L^n, {n, 0, m}]
F[z_] = Sqrt[3/z] ( 1 + Sum[P[m, Log[z]]/z^m, {m, 1, M}])
M = 9; a[1, 0] = 0;
F1x = F[1 + 1/x];
Ftx = Sin[F[1/x]];
s[1] = Series[(F1x - Ftx)/Sqrt[x], {x, 0, 2}]
t[1] = Extract[Solve [Coefficient[s[1], x^2] == 0, {a[1, 1]}], 1]
A[1, 1] = ReplaceAll[a[1, 1], t[1]]
su[1] = t[1]
```

```
m=2; s[m]=Simplify[ReplaceAll[Series[(F1x-Ftx)/Sqrt[3x], {x, 0, m+1}], su[m-1]]]
t[m] = Simplify[Coefficient[ReplaceAll[s[m], Log[x] -> L], x^(m+1)]]
u[m] = Simplify[Collect[t[m], L]]
v[m] = Table[Coefficient[u[m] L, L^(n+1)] == 0, {n, 0, m}]
w[m] = Table[a[m, n], {n, 0, m}]
ad[m] = Extract[Solve[v[m], w[m]], 1]
su[m] = Join[su[m-1], ad[m]]

m=3; s[m]=Simplify[ReplaceAll[Series[(F1x-Ftx)/Sqrt[3x],{x, 0, m+1}], su[m-1]]]
t[m] = Simplify[Coefficient[ReplaceAll[s[m], Log[x] -> L], x^(m+1)]]
u[m] = Simplify[Collect[t[m], L]]
v[m] = Table[Coefficient[u[m] L, L^(n+1)] == 0, {n, 0, m}]
w[m] = Table[a[m, n], {n, 0, m}]
ad[m] = Extract[Solve[v[m], w[m]], 1]
su[m] = Join[su[m-1], ad[m]]
```

Table 1. Coefficients *a* in equation (8)

1	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	$a_{0,5}$	$a_{0,6}$	$a_{0,7}$	$a_{0,8}$
0	$-\frac{3}{10}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$	$a_{1,5}$	$a_{1,6}$	$a_{1,7}$	$a_{1,8}$
$\frac{79}{700}$	$-\frac{9}{50}$	$\frac{27}{200}$	$a_{2,3}$	$a_{2,4}$	$a_{2,5}$	$a_{2,6}$	$a_{2,7}$	$a_{2,8}$
$\frac{411}{3500}$	$-\frac{1941}{7000}$	$\frac{27}{125}$	$-\frac{27}{400}$	$a_{3,4}$	$a_{3,5}$	$a_{3,6}$	$a_{3,7}$	$a_{3,8}$
$\frac{1606257}{10780000}$	$-\frac{7227}{17500}$	$\frac{1683}{4000}$	$-\frac{1917}{10000}$	$\frac{567}{16000}$	$a_{4,5}$	$a_{4,6}$	$a_{4,7}$	$a_{4,8}$
$\frac{140345627}{700700000}$	$-\frac{70079931}{107800000}$	$\frac{566973}{700000}$	$-\frac{98739}{200000}$	$\frac{7533}{50000}$	$-\frac{15309}{800000}$	$a_{5,6}$	$a_{5,7}$	$a_{5,8}$
$\frac{137678711441}{490490000000}$	$-\frac{7364523}{7007000}$	$\frac{305491257}{196000000}$	$-\frac{4155111}{3500000}$	$\frac{796311}{1600000}$	$-\frac{2218347}{20000000}$	$\frac{168399}{16000000}$	$a_{6,7}$	$a_{6,8}$
$\frac{25317035192599}{62537475000000}$	$-\frac{8462569406199}{4904900000000}$	$\frac{32174780481}{10780000000}$	$-\frac{5367503637}{1960000000}$	$\frac{407711313}{280000000}$	$-\frac{181900809}{400000000}$	$\frac{1960281}{25000000}$	$-\frac{938223}{160000000}$	$a_{7,8}$
$\frac{20390161579066727223}{34157135012000000000}$	$-\frac{1191102095494119}{416916500000000}$	$\frac{111297984941043}{19619600000000}$	$-\frac{164937331809}{26950000000}$	$\frac{310033164249}{78400000000}$	$-\frac{11083463037}{7000000000}$	$\frac{3104886087}{8000000000}$	$-\frac{301983147}{5600000000}$	$\frac{8444007}{2560000000}$

and so on. In order to simplify the tracing of the algorithm above, no loop with respect to *m* is arranged. The resulting coefficients are shown in Table 1.

For some fixed integer $M > 1$, define function F as the limit

$$F(z) = \lim_{k\to\infty} \arcsin^k\left(F_M(z+k)\right). \tag{9}$$

While function F by equation (6) is asymptotic solution of the transfer equation, function F does not depend on the number M of terms taken into account. However, for larger M, the limit converges faster, and this is important for the efficient numerical implementation. Then, through function F, the super sin can be defined as follows:

$$\mathrm{SuSin}(z) = F(z + x_1), \tag{10}$$

where $x_1 \approx 1.4304553465288$ is solution of equation $F(1 + x_1) = 1.$ This condition determines that

$$\mathrm{SuSin}(1) = 1 \tag{11}$$

and

$$\mathrm{SuSin}(0) = \arcsin(1) = \pi/2. \tag{12}$$

Then $\mathrm{SuSin}(n) = \sin^n(\pi/2)$ can be interpreted as the result of *n* applications of function sin to initial argument $\pi/2,$ giving sense to this expression for non-integer and even non-real (complex) number *n* of iteration.

The numerical algorithm for evaluation of super sin by equation (10) is implemented in C++ with complex double variables. Chosen value $M = 8$ corresponds to 9 terms (counting the zeroth leading term), taken into account in equation for F_M. Of order of 40 iterations of arcsin are used for evaluation of SuSin of argument of order of unity. This algorithm is used to plot figures. The accuracy of the resulting algorithm is estimated to be of order of 14 decimal digits, and it is close to maximal precision, achievable for the implementation with the complex double variables. This precision should be compared to 7 decimal figures, achievable with several thousand iterations, reported by [17] for only one leading term taken into account.

Graphic $y = \mathrm{SuSin}(x)$ is shown in the top part of Figure 1 with thick line. For comparison, the asymptotic $y = \sqrt{3/x},$ valid for large values of *x*, is shown with upper thin line. The leading term of the expansion at zero, discussed below, is shown with lower thin line.

Complex map of function SuSin is shown in the bottom part of Figure 1 with lines $u = \Re(\mathrm{SuSin}(x + \mathrm{i}y))$ and lines $v = \Re(\mathrm{SuSin}(x + \mathrm{i}y))$ in the *x*, *y* plane. The cut of range of holomorphism along the negative part of the real axis is marked with dashed line.

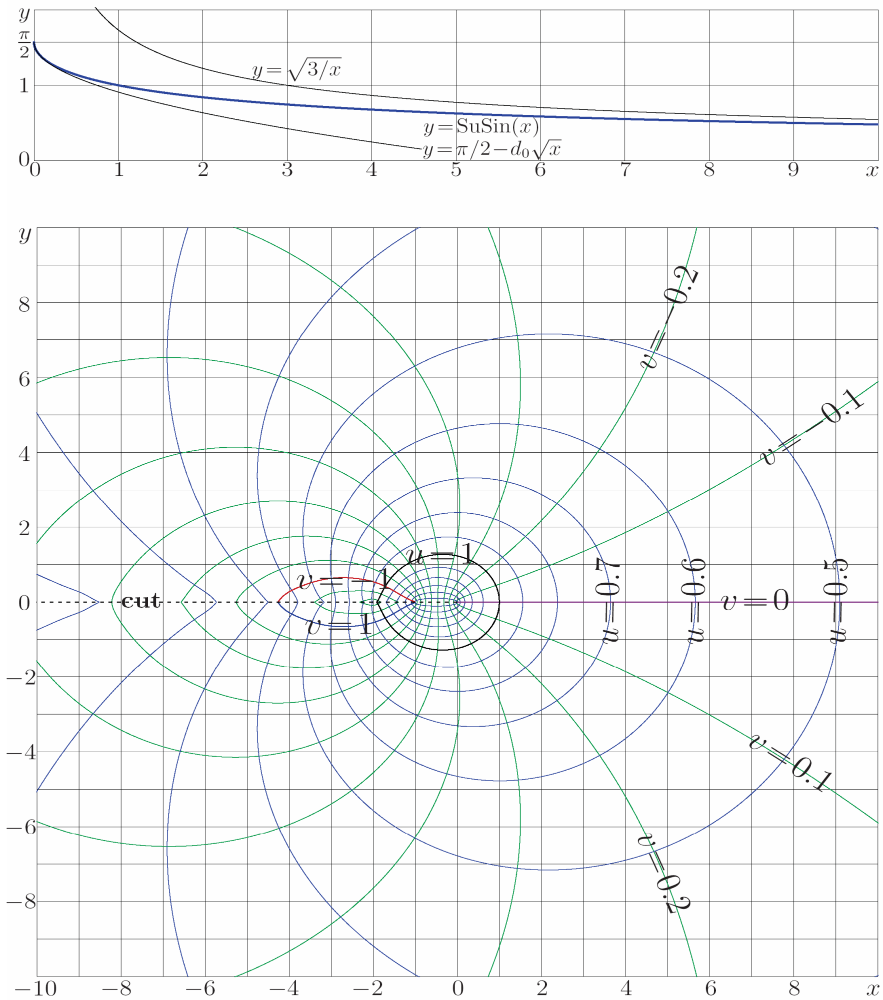

Figure 1. Top: $y = \mathrm{SuSin}(x)$ by (10), thick intermediate curve; $y = \sqrt{3/x}$, upper curve; $y = \pi/2 - d_0\sqrt{x}$, lower curve; Bottom: complex map of SuSin by (10): $u + \mathrm{i}v = \mathrm{SuSin}(x + \mathrm{i}y)$ in the x, y plane.

Function SuSin has the root singularity at zero. Expansion at the singularity can be written as follows:

$$\mathrm{SuSin}(z) = \frac{\pi}{2} - \sqrt{z}\sum_{m=0}^{M} d_m z^m + O(z^{M+3/2}). \tag{13}$$

The coefficients d of expansion (13) can be evaluated with the Cauchi integral, calculating the Taylor expansion of function $z \mapsto -\sqrt{z}(\text{SuTra}(z) - \pi/2)$. The approximations for coefficients d are:

$$d_0 \approx 0.66058238000676,$$

$$d_1 \approx -0.12329860399822,$$

$$d_2 \approx 0.05094582111508,$$

$$d_3 \approx -0.02828892576497,$$

$$d_4 \approx 0.01839521673418,$$

$$d_5 \approx -0.01316131268331.$$

The truncated expansion by (13) with $M = 0$ is also shown in Figure 1 with the lowest thin curve.

Function SuSin can be approximated also with simple function

$$\text{SuSin}(z) \approx \exp((1 - \sqrt{z}) \ln(\pi/2)). \tag{14}$$

The approximation by (14) is valid for moderate values of z; while $|z| < 1.5$, it returns two correct decimal digits. This approximation had been suggested by Curtright and Zachos [13] at

http://server.physics.miami.edu/~curtright/Schroeder.html.

That approximation allows to draw the explicit plot of function SuSin of real argument and iterates of function sin.

Function SuSin, constructed in this section, is super sin, declared in the title of the article. Up to my knowledge, formulas (7)-(10) provide the most efficient (fast and precise) algorithm for evaluation of SuSin, among ever reported in the literature. For evaluation of non-integer iterates, the inverse function is also required, i.e., the Abel sin. In the next section, it is denoted with $\text{AuSin} = \text{SuSin}^{-1}$.

4. Abel Sin

This section describes function $\mathrm{AuSin} = \mathrm{SuSin}^{-1}$, which is Abel function for sin. The explicit plot of this function is shown in Figure 2. Evaluation of this function is described below.

AuSin satisfies the Abel equation

$$G(\sin(z)) = G(z) + 1 \tag{15}$$

which is just equation (2) at $T = \sin$. Construction of the asymptotic expansion for AuSin is similar to that of SuSin. First, some solution G is constructed with leading term of the asymptotic expansion

$$G(z) = \frac{3}{z^2} + O(\ln(z)) \tag{16}$$

which corresponds to function SuSin^{-1}. Then the constant is added to satisfy the additional condition $\mathrm{AuSin}(1) = 1$. While the asymptotic of G corresponds to asymptotic of function F, the conjecture is that $F = G^{-1}$ and $\mathrm{SuSin} = \mathrm{AuSin}^{-1}$.

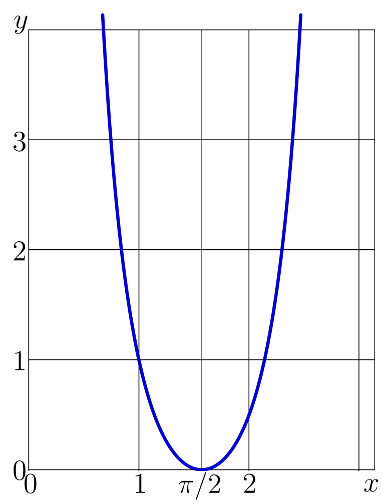

Figure 2. $y = \mathrm{AuSin}(x)$ by (20).

Let

$$G_M(z) = \frac{3}{z^2} + \frac{5}{6}\ln(z) + \sum_{m=1}^{M} c_m z^{2m}. \tag{17}$$

Substitution of $g(z) = G_M(z) + O(z^{2M+2})$ into the Abel equation (15) gives the coefficients c. The automatic calculation of the coefficients c is straightforward and even simpler, than calculation of coefficients a in expansion (7), (8); so, I do not copy paste the Mathematica code here. The first 8 resulting coefficients are

$$\begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \\ \frac{79}{1050} & \frac{29}{2625} & \frac{91543}{36382500} & \frac{18222899}{28378350000} & \frac{88627739}{573024375000} \end{array}$$

$$\begin{array}{ccc} c_6 & c_7 & c_8 \\ \frac{3899439883}{142468185234375} & \frac{32544553328689}{116721334798818750000} & \frac{4104258052789}{15547297342500000000} \end{array}. \tag{18}$$

Then function G can be evaluated with

$$G(z) = \lim_{k\to\infty} G_M(\sin^k(z)) - k. \tag{19}$$

While G_M is asymptotic solution of the Abel equation with sin as the transfer function, the limit does not depend on M. However, the rate of convergence improves with the increase of M.

Through function G, by (19), the Abel sin appears as

$$\mathrm{AuSin}(z) = G(z) - G(\pi/2) \tag{20}$$

in such a way that $\mathrm{AuSin}\left(\frac{\pi}{2}\right) = 0$. Constant $G(\pi/2) \approx 2.089622719729524$.

The numerical implementation of AuSin is loaded as http://mizugadro.mydns.jp/t/index.php/ausin.cin. This algorithm is used to plot figures. The explicit plot of AuSin is shown in Figure 2. The complex map of AuSin is shown in Figure 3 in the same way, as map of SuSin is shown in Figure 1;

$u + \mathrm{i}v = \mathrm{AuSin}(x + \mathrm{i}y)$. The limit in (19) converges quickly (within a hundred iterations) at least within the range $|z - \pi/2| < \pi/2$; outside, some fractal behaviour is seen in the figure.

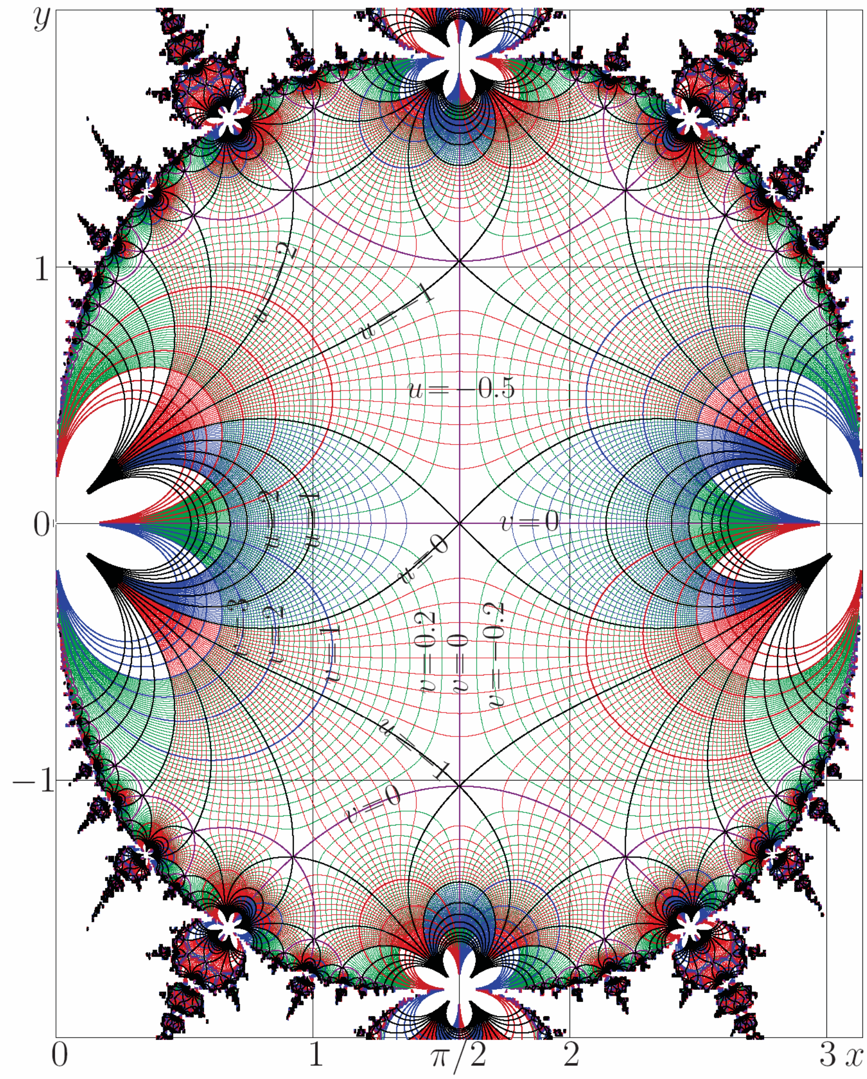

Figure 3. $u + \mathrm{i}v = \mathrm{AuSin}(x + \mathrm{i}y)$ by (20).

The range of validity of relation

$$\mathrm{SuSin}(\mathrm{AuSin}(z)) = z \tag{21}$$

is shown in Figure 4 with dense grid. The dense grid is formed with the complex map of the left hand side of equation (21).

The inverse relation

$$\mathrm{AuSin}(\mathrm{SuSin}(z)) = z \tag{22}$$

is valid in the whole complex plane except the halfline $z < 0$.

The boundary of the domain of validity of equation (21) follows the lines $\Im(\mathrm{AuTra}(z)) = 0$. These lines are also shown in Figure 4, they are borrowed from Figure 3.

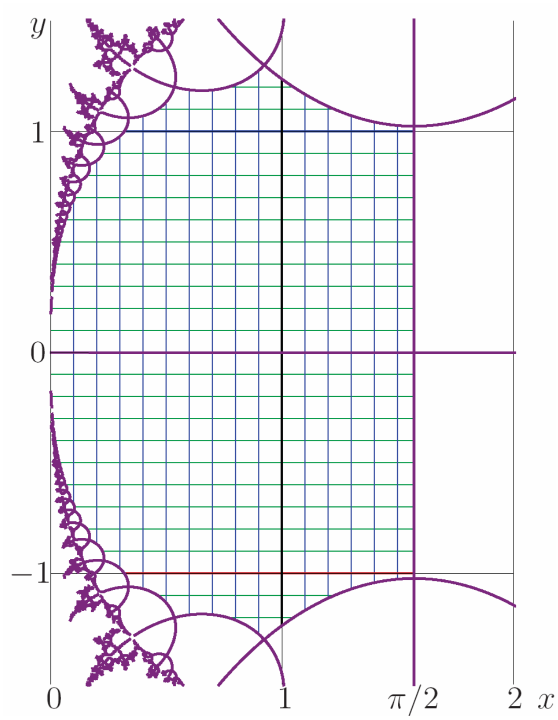

Figure 4. Range of (21) in $z = x + \mathrm{i}y$ plane.

Within the range shaded with dense rectangular grid in Figure 4, the group relation

$$\sin^m(\sin^n(z)) = \sin^{m+n}(z) \tag{23}$$

holds at least for real m and n. The numerical implementation mentioned reproduces it with at least 14 decimal figures. In particular, relation 23 holds in vicinity of the real axis.

The precise implementation of AuSin for complex argument allows to calculate the Taylor expansion at $\pi/2$,

$$\mathrm{AuSin}(z) = \sum_{n=1}^{\infty} b_n (z - \pi/2)^{2n}. \tag{24}$$

The series converges while $|z - \pi/2| < \pi/2$. The coefficients b of this expansion can be evaluated through the Cauchi integral. Approximations for first 6 coefficients in (24) are

$$b_1 \approx 2.29163807440958, \tag{25}$$

$$b_2 \approx 1.96043852439688, \tag{26}$$

$$b_3 \approx 1.07862851256147, \tag{27}$$

$$b_4 \approx 0.59622997993395, \tag{28}$$

$$b_5 \approx 0.28333997139829, \tag{29}$$

$$b_6 \approx 0.14193261194548. \tag{30}$$

Function AuSin has sense of number of iterates of sin, beginning with $\pi/2$, required in order to get value of argument. However, the number of these iterates has no need to be integer. With the expansion (17) at zero and the Taylor expansion (24), and the Abel equation (15), function AuSin can be evaluated within few tens operations with 14 decimal figures. Then, with functions SuSin and AuSin, the non-integer iterates of function sin can be expressed also for other initial values of the argument (it has no need to be $\pi/2$ and may have complex value). This is considered in the next section.

5. Iterates of Sin

While functions SuSin and AuSin are defined and described, the nth iterate of sin can be defined as

$$\sin^n(z) = \mathrm{SuSin}(n + \mathrm{AuSin}(x)). \tag{31}$$

For real values of argument, the iterates of sin are shown in Figure 5; $y = \sin^n(x)$ is plotted versus x for various values n. Curves, that correspond to integer values of n, are thick; the thin curves correspond to non-integer values of number n of iterates. Similar curves can be plotted with the approximation of SuSin suggested by [13].

In Figure 5, curve for $n = 1$ is just $y = \sin(x)$, and that for $n = -1$ is just $y = \arcsin(x)$. In such a way, the non-integer iterates allow the smooth (holomorphic) transition from a function to its inverse function. With the efficient algorithms for SuSin and AuSin, described above, all the figures can be generated in real time; these functions can be used as other special functions.

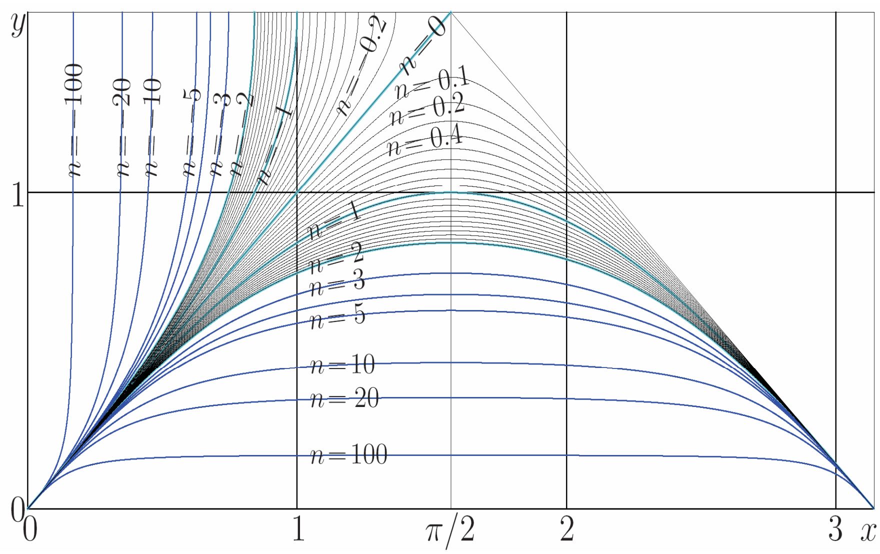

Figure 5. $y = \sin^n(x)$ versus x for various values of number n of iterate.

6. Discussion

Iterations of sin may have various applications. The high iterate of sin, say, $\sin^{100}$, may describe the shape of sledge runners. Figure 6 shows the approximation of the sled runner from photo

http://en.wikipedia.org/wiki/File:Boy_on_snow_sled,_1945.jpg

by [2]. The approximating curve is

$$y = \sin^n(\pi/2) - \sin^n(x) \tag{32}$$

at $n = 100$. This number is only adjusting parameter in the fitting. The curve in Figure 6 corresponds to the lowest curve in Figure 5, flipped upside-down.

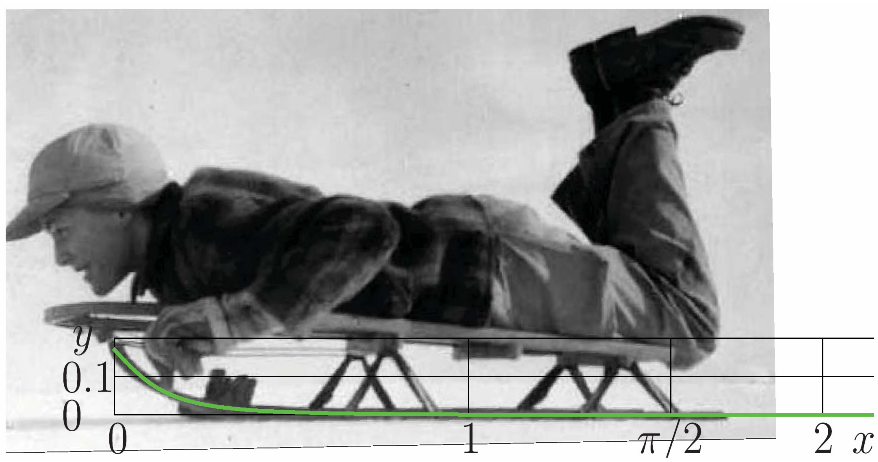

Figure 6. Approximation of shape of the sledge runner by [2] with iterate of sin, $y = \sin^n(\pi/2) - \sin^n(x)$ with single adjusting parameter $n = 100$.

Fitting by (32) is the best single-parametric approximation I could suggest for this case. In this application, there is no serious reason for n to be integer; it could be some real number instead. This is one of the reasons, why the evaluation of non-integer iterates has sense.

The skeptics may say, that at large $n >> 1$, iteration $\sin^n$ can be approximated with its asymptotic

$$\sin^n(x) \approx \sqrt{\frac{3}{n + 3/x^2}}. \tag{33}$$

Then one can replace the constants in expression (33) to parameters and even improve the fitting. However, the result will not be a single-parametric fit.

In formalism of superfunctions, sin is simple example of a transcendent function that has unity derivative and zero second derivative at the fixed point. For this case, the superfunction cannot be constructed as the expansion with exponentials, as it can be done for the exponential to base b between 1 and $\exp(1/e)$ [10], for factorial [11], and for many other transfer functions. Sin is widely used function, so, its non-integer iterates should be considered as special functions too.

7. Conclusion

The super sin function named SuSin is defined with equation (10) as solution of the transfer equation (5) with sin as the transfer function. Expansions of this function at zero and at infinity are suggested. The complex double implementation in C++ is loaded at

http://mizugadro.mydns.jp/t/index.php/susin.cin;

the complex map of SuSin is shown in Figure 1. The algorithm suggested is significantly more efficient than the approximations reported recently [17, 18].

The Abel sin function named $\mathrm{AuSin} = \mathrm{SuSin}^{-1}$ is constructed with equation (20). This function is solution of the Abel equation (15). The complex double implementation in C++ is loaded at

http://mizugadro.mydns.jp/t/index.php/ausin.cin;

the complex map of AuSin is shown in Figure 3.

With SuSin and AuSin, iterates of function sin can be expressed with equation (31); the iterate has the group property $\sin^m(\sin^n(z)) = \sin^{m+n}(z)$. The range of validity of this representation is shown in Figure 4. For real values of the argument, the iterates of sin are shown in Figure 5. Up to my knowledge, this figure represents the most precise evaluation of non-integer iterates of sin, ever reported.

In general, superfunctions, Abel functions and the resulting non-integer iterates of special functions greatly extend the arsenal of holomorphic functions, available for the description of physical processes. For this reason, superfunctions for the elementary functions should be described and implemented in the programming language. The algorithms described here, as well as those presented recently [8-12, 15, 21], can be used as prototypes for the implementation of superfunctions, Abel functions and iterates of special functions in both the commercial and the free software. In particular, super sin and iterates of sin should get status of special functions.

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