Nemtsov function and its iterates

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Abstract The Nemtsov function appears as polynomial $\mathrm{Nem}_q(z) = z + z^3 + qz^4$; q is parameter. The Superfunction, Abelfunction and iterates Nem_q^n for complex n are constructed.

Keywords Nemtsov function, Superfunction, Abelfunction, Iterate

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1 Preface

This article describes properties of the special 4th order polynomial

$$\mathrm{Nem}_q(z) = z + z^3 + qz^4 \tag{1}$$

where $q \geq 0$ is parameter. Explicit plot of this function is shown in figure 1 for various values of q. The complex map is shown in figure 2 for $q=0$ and for $q=2$.

Function Nem by (1) is considered as an attempt to reject (or to correct, if necessary) the pretentious claim below:

Claim 1.
Any growing real-holomorhic function T can be declared as "transfer function".
Then, for this transfer function, the real-holomorphic superfunction F, the abelfunction $G=F^{-1}$ and the iterates $T^n = F\big(n+G(z)\big)$ can be defined and implemented.
The only boundaries of the range of holomorphism, id est, the cut lines in the complex plane, limit validity of this representation.

Superfunction F appears as solution of the transfer equation

$$T(F(z)) = F(z+1) \tag{2}$$

with appropriate additional requirements, that should provide uniqueness of the solution F.

The corresponding abelfunction $G = F^{-1}$ satisfies the Abel equation

$$G(T(z)) = G(z) + 1 \tag{3}$$

Equation (3) (with additional conditions necessary for the uniqueness) also can be considered as "primary" (then, superfunction F appear as inverse of abelfunction G). For the evaluation of F and G, both equations (2) and (3) are useful.

Once, for some transfer function T, the superfunction F and the abelfunction G are established, the iterates of the transfer function T can be expressed as follows:

$$T^n = F\big(n+G(z)\big) \tag{4}$$

Here, the superscript after the name of function indicates the number of its iterate; past century this notation had been suggested by W.Bergweiler [4]. In these notations,

$$T^0(z) = z \tag{5}$$
$$T^1(z) = T(z) \tag{6}$$
$$T^2(z) = T(T(z)) \tag{7}$$
$$T^3(z) = T(T(T(z))) \tag{8}$$

and so on; in particular, $\sin^2(x)$ denotes $\sin(\sin(x))$, but neither $\sin(x)^2$ nor $\sin(x^2)$.

In representation (4), number n of iterate has no need to be integer. In particular, it can be a complex number. Iterates appear especially explicit for a growing real-holomorphic transfer function; however, other holomorphic functions also can be treated in the similar way.

Many examples of special transfer functions T are already considered, including exp to various bases, quadratic polynomials, factorial and sin. Overview of these results is presented in the Introduction below.

Consideration of the examples is important for testing of Claim 1, in order to see, if it should be formulated better. The goal is to convert it to as mathematical conjecture, and then - to a theorem, supplying it with the rigorous proof.

Before to search for the correct rigorous formula (and rigorous proof) of some statement, it worth to check it with examples, trying to reject it. Now, the work with superfinction is at stage of attempts to reject, to refute Claim 1 above. While, all the attempts to refute Claim 1, fail: for each transfer function T considered, at least one superfunction F and the corresponding abelfunction G had been constructed. This arcticle shows, that function $T = \mathrm{Nem}_q$ by (1) is not exception.

The 4th order polynomial (1) is especially interesting for the following reasons.
1. Its inverse function can be expressed as combination of elementary functions.
2. For such a transfer function, the previously reported methods of construction of superfunctions and abelfunctions fail, as the expansion at the fixed point begins with

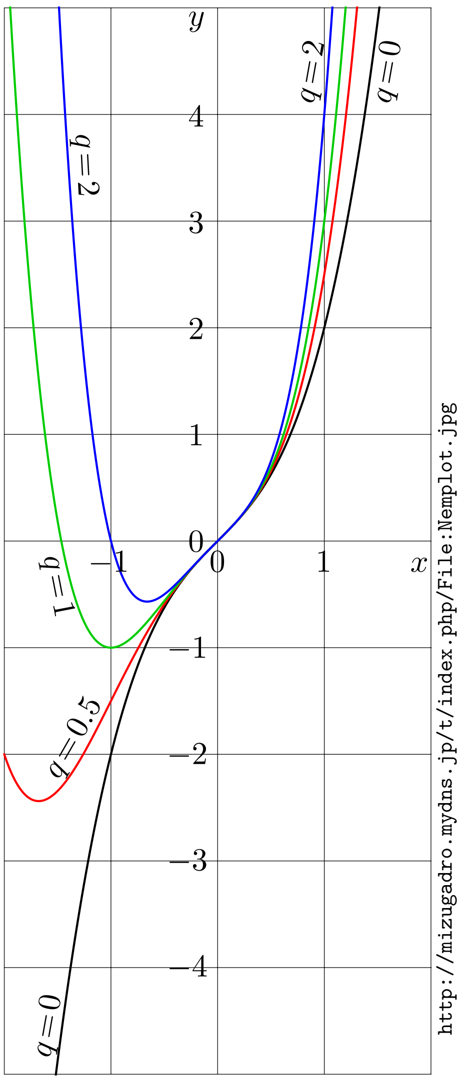

Figure 1. $y = \mathrm{Nem}_q(x)$

identity function, cubic term and 4th order term.
3. The methods developed below for this polynomial, can be used for other functions with similar expansion at the fixed point.

The need to assign a special name the 4th order polynomial by (1) had been revealed, realised, formulated 2016.02.27, and I begun to search for an appropriate name for it. Next day, the horrible news appeared, that Putin had killed his opponent Boris Nemtsov. Evidences, why the only Putin could do this, fall out of scope of this article; but the last name of the victim happened to be strongly connected to polynomial (1). So, I call it "The Nemtsov function", and denote it with three-letter abbreviation "Nem", indicating parameter q in subscript.

In such a way, this article is dedicated to Boris Nemtsov, although he, perhaps, never knew about importance of the specific 4th order polynomial (1), mentioned in the Abstract and called after his name.

Analysis of the Nemtsov function appears as an attempt to construct a growing holomorphic function, such that its superfunction, abelfunction and the non-integer iterates cannot be constructed with methods already described in literature. Partially, this goal is achieved: The formalism [15] described for the similar function sin, needs to be a little bit generalised to treat the Nemtsov function in a similar way. Here, term "similar" indicates similarity in the expansion of the transfer function at its fixed point.

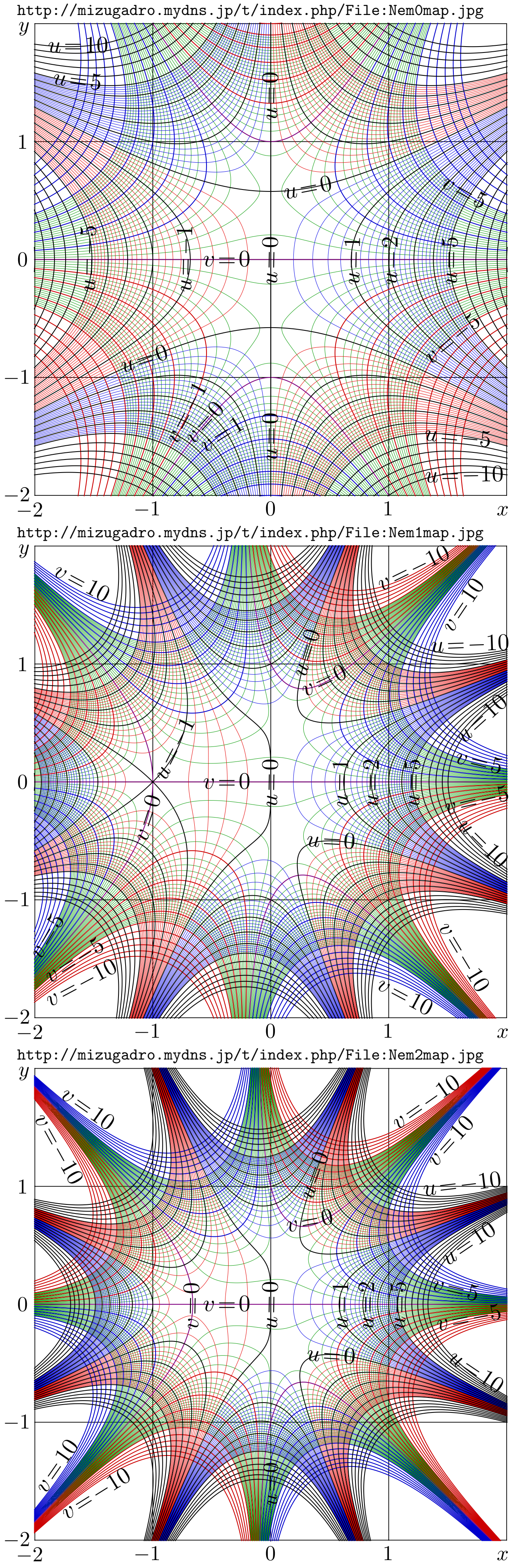

Figure 2. Maps $u+\mathrm{i}v = \mathrm{Nem}_q(x+\mathrm{i}y)$ for $q=0,1,2$

In this article, properties of the Nemtsov function Nem by equation (1) and its inverse function ArqNem = Nem^{-1} are discussed.
Function $T=\mathrm{Nem}_q$ is treated as a transfer function.
Superfunction $F = \mathrm{SuNem}_q$ and abelfunction $G = \mathrm{AuNem}_q = \mathrm{SuNem}_q^{-1}$ are constructed as solutions of equations (2) and (3).
Then, the iterates Nem_q^n by (4) are described.
The construction may be used also for other functions T, not necessary polynomial, while the expansion at the fixed point begins with polynomial (1).

The analysis mentioned is important as an approach to the more pretentious objective. The challenging goal is construction of an automatic algorithm, procedure, for evaluation of iterates of holomorphic function of general kind.

Ideally, the iterates should be performed in a similar way, as the high level algorithmic languages evaluate automatically derivatives and integrals of special functions. In particular, the Mathematica function Nest[x,F,n] should not require, that the number n of iterates is expressed with integer number (as it requires in the current versions of Mathematica). The procedure should chose the appropriate fixed point(s) L (if equation $T(L)=L$ has a solution), guess the as asymptotic of superfunction F and abelfunction G, and take into account, that the first derivative at the fixed point may happen to be unity, and the second derivative may happen to be zero (as it takes place for the Nemtsov function). The consideration below gives a hint, now to approach such a great and challenging objective.

2 Introduction

In this section I mention some recent results, to release the colleagues from the need to drill the previous publications on the topic.

2.1 Overview

Since 1950, the interest to non-integer iterates is boiled-up with iterates of exponential and, in particular, iterate half [1], id est, function φ such that $\varphi(\varphi(z))=\exp(z)$. The problem of the regular iteration [2, 3, 4, 5] of holomorphic function had been formulated, although until year 2009, no efficient algorithm for computation of non-integer iterates (except few special functions) had been suggested. Then, such algorithms had been reported; and not only for the exponent to various bases [6, 7, 8, 9, 14, 18, 19, 20] but also for other holomorphic functions: for factorial by [10], for the "logistic operator" $z\mapsto s\,z\,(1-z)$ by [11], for the "Trappmann function" $z\mapsto z+\exp(z)$ and function $z\mapsto z\exp(z)$ by [13], and for sin by [15].

At calculation of the non-integer iterates of some transfer function T, its superfunction F and abelfunction G, the important issue is about the fixed points of the transfer function, id est, solutions L of equation $T(L) = L$. The important case refers to the real-holomorphic function T with real fixed point L and positive (but not unity) derivative in this point, id est, $T'(L)>0$, $T'(L)\neq 1$. Then, the first non-zero term in the expansion of T at the fixed point L determines the asymptotic behaviour of the superfunction F; this gives way of the efficient evaluation. The basic formula for this case are mentioned below.

2.2 Regular iteration: $T'(L)>0$, $T'(L)\neq 1$

Case $T'(L)>0$, $T'(L)\neq 1$ is qualified as "Regular iteration". The asymptotic solution of the superfunction can be written as follows:

$$F(z) = L + \sum_{n=1}^{N} a_n \varepsilon^n + o(\varepsilon^N) \qquad (9)$$

where
N is natural number,
$\varepsilon = \exp(kz)$ for some constant k and
a are constant coefficients.

The substitution into the [[transfer equation]] and the asymptotic analysis at $\varepsilon \ll 1$ gives

$$k = \log(K) = \log(T'(L)) \qquad (10)$$

and set of equations for coefficients a. It is convenient to set $a_1 = 1$. Then, other coefficients appear as solutions of equations

$$a_2K - a_2K^2 + T''(L)/2 = 0 \qquad (11)$$

$$a_3K - a_3K^2 + 2a_2T''(L)/2 + T'''(L)/6 = 0 \qquad (12)$$

and so on.

With the truncated asymptotic representation $\tilde F$, the superfunction F appears as limit,

$$F(z) = \lim_{m\to\pm\infty} T^m(\tilde F(-m+z)) \qquad (13)$$

where sign + or - is chosen dependently on $T'(L)$, in order to provide smallness of ε and validity of the expansion at large positive or at large negative m.

The resulting iterates happen to be regular at the fixed point L; to, this case is called "regular iteration". Such a method allows to construct super exponential to base between unity and $\exp(1/\mathrm{e}) = \exp^2(-1)$, super factorial, and other superfunctions. Even supertetration (id est, pentation) can be constructed in this way. [8, 10, 11, 14].

The expansion above and the resulting solution are not valid at $K = \log(T'(L)) = 0$, id est, af $T'(L) = 1$. For analysis of Claim 1 in the Preface, this case should be considered.

2.3 Exotic iteration: $T'(L)=1$, $T''(L)\neq 0$

Assume, $T'(L) > 0$, $T'(L) = 1$, but the second derivative of the transfer function at the fixed point is not zero, $T''(L) \neq 0$. In particular, this refers to $T = \exp_\eta$, with specific base $\eta = \exp(-1/\mathrm{e})$, see [9].

For this case, the asymptotic expansion of superfunction happens to be a little bit more complicated:

$$F(z) \sim L + \frac{a}{z} + \sum_{m=1}^{M} \frac{P_m(\ln(\pm z))}{z^{m+1}} + O\left(\frac{\ln(\pm z)^{M+1}}{z^{M+1}}\right) \qquad (14)$$

Here, $a=-2/T''(L)$ is constant, and P_m is polynomial of m-th order. Coefficients of these polynomials can be calculated, substituting the asymptotic representation (14) into the transfer equation (2). Sign + or - in the argument pf ;pgarothm should be chosen, dependently, should the iterate $T^n(z)$ be holomorphic at $z > L$ or at $z < L$. In general cases, it seems to be impossible to realise both at once. At fixed point L, the non-integer iterate has the branch point. For this reason, as oppose to the "regular iteration", this case is qualified as "exotic". In order to get iterates $T^n(z)$, growing at $z > L$, we should choose sign $-$. Then, truncation of the asymptotic expansion at some M gives the approximation, valid for large negative values of the real part of the argument. From these values, the superfunction can be evaluated with any required precision. Then, the solution can be extended to the whole complex plane by limit (13).

In such a way, the asymptotic solution determines (and gives way for the efficient evaluation) of the superfunction.

In such a way, for this method, condition $T''(L) \neq 0$ is essential. Again, in support of Claim 1, such a case is of interest. Superfunction for tetration, id est, supertetratin, denoted as "pentation", also can be constructed in this way [14].

2.4 Exotic: $T'(L)=1$, $T''(L)=0$, $T'''(L)\neq 0$

Case $T'(L)=1$, $T''(L)=0$, $T'''(L)\neq 0$ is considered for the case $T(-z)=-T(z)$; symmetry simplifies consideration [15]. The expansion appears as follows

$$F(z) \sim L + \frac{a}{z} + \sum_{m=1}^{M} \frac{P_m(\ln(\pm z))}{z^{m+1}} + O\left(\frac{\ln(\pm z)^{M+1}}{z^{M+1}}\right) \quad (15)$$

where P_m is some polynomial; substitution of the expansion into the transfer equation determines coefficients of this polynomialal. This method is tested with $T = \sin$; the superfunction $F = \mathrm{SuSin}$, the abeldunction $G = \mathrm{AuSin}$ and the iterates $\sin^n$ are constricted [15].

Expansion sina is valid for a symmetric transfer function $T(-z)=-T(z)$, as it takes place for $T = \sin$.

So, the method [15] cannot be applied functions without this symmetry; in general case, $T(-z)$ has no need to be equal to $-T(z)$.

Searching for the generalisation, I consider another example, when such a symmetry is broken. Looking for a simplest example of such a function, I choose the specific polynomial Nem_q by (1). This is why, consideration of the Nemtsov function serves as test of Claim 1, formulated in the Preface.

3 Preliminary analysis

For some transfer functions, the superfunction, abelfunction and the non-integer iterates can be expressed through the special functions. In particular, this applies to the power function. Consider transfer function

$$T(z) = cz^{1+r} \quad (16)$$

The superfunction F, abelfunction G and iterates T^n appear as follows:

$$F(z) = c^{-1/r} \exp\left((1+r)^z \, c^{1/r}\right) \quad (17)$$

$$G(z) = \frac{\ln\left(c^{-1/r}\, \ln(c^{1/r}\, z)\right)}{\ln(1+r)} \quad (18)$$

These functions can be verified with Mathematica code

```
F[z_] = Exp[c^(1/r)*(1+r)^z]/c^(1/r)
G[z_] = Log[Log[c^(1/r)*z]/c^(1/r)]/Log[1+r]
Simplify[F[1+G[z]], {r>0,c>0}]
```

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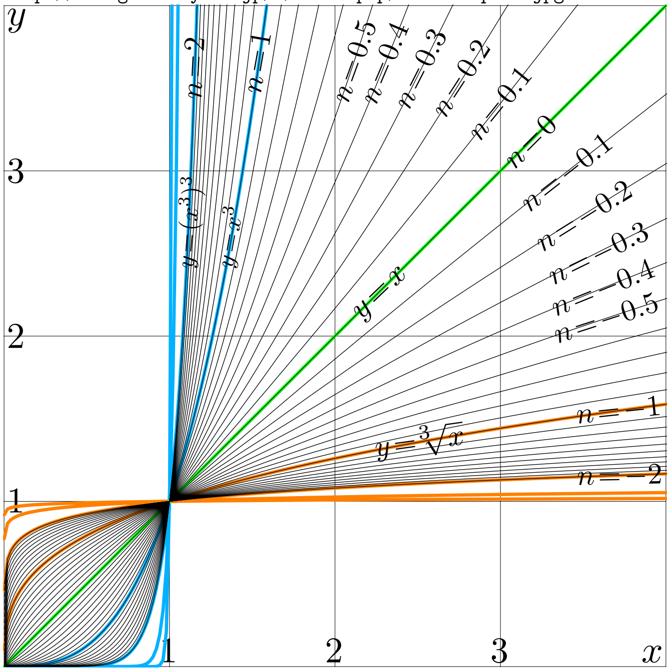

Figure 3. $y = T^n(x) = x^{3^n}$ by (19)

The output is $\mathrm{c\,z}^{1+\mathrm{r}}$

For case $c=1$, $r=2$, iterates

$$T^n(z) = z^{3^n} \quad (19)$$

are shown in figure 3. All the curves in this figure pass through the fixed points of the transfer function, id est, zero and unity. In the similar way, one may construct the complex maps of the iterates, using either the explicit representation (19), or general formula $T^n(z) = F(n + G(z))$. And, of course not only for $c = 1$, $r = 2$, but for other values too.

Also, one can imagine, how should look the maps of iterates of the Nemthsov function. At large values of the argument, they should look similar to maps of iterates of the power function; it est, similar, to the maps of an appropriate power function, as iterate of the power function is also a power function. An example of such a map is shown in figure 4.

The beautiful maps for iterates of a power function provokes a question: Is it possible to construct the similar iterates for the Nemtsov function, $T = \mathrm{Nem}_q$?

The answer is "yes". Such iterates are constructed and described below.

4 Inverse function

For the non-integer iterate of some function, the inverse function is necessary. Then, at least set of integer iterates is covered. This section describes function $\mathrm{ArqNem}_q = \mathrm{Nem}_q^{-1}$; its maps for $q\to 0$, for $q=1$ and $q=2$ are shown in figure 5.

The appropriate choice of the cut lines of the inverse function happens to be the important part of this research. Complex maps of function ArqNem_q are shown in figure 5 for the same values of q, as maps of function Nem_q in figure 2. Construction of function ArqNem_q is described below.

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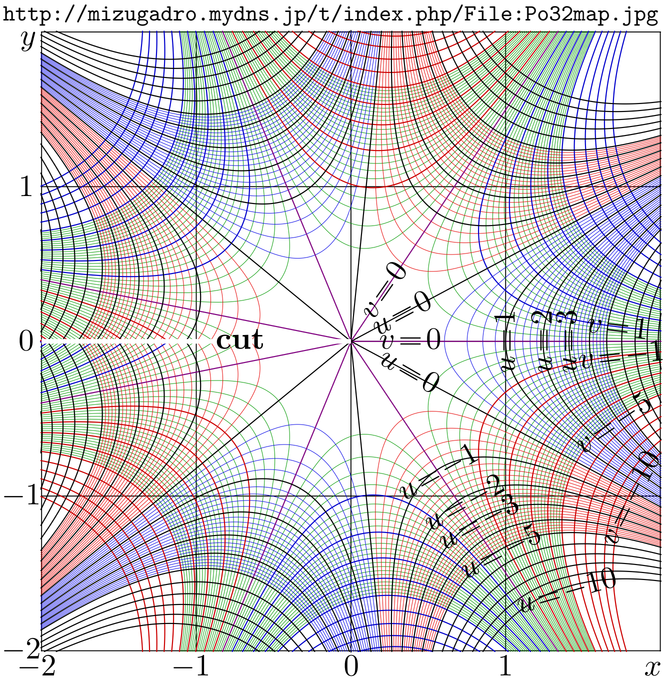

Figure 4. $u+\mathrm{i}v=(x+\mathrm{i}y)^{3.2}$

Solutions of the 4th order algebraic equation can be expressed with the Mathematica code

```
T[z_] = z + z^3 + q z^4;  Solve[T[f]==z,f]
```

Formally, the result is correct, but it is cumbersome and not so good for plotting of complex maps: The cut lines of any of the four solutions divide the complex plane to barely connected domains. In order to handle the cuts, we need to evaluate the branch points of function Nem_q^{-1}, The branch points appear in form $\mathrm{Nem}_q(s)$, where s is solution of equation $\mathrm{Nem}_q'(s)=0$,

```
T1[z_] = 1 + 3 z^2 + 4 q z^3;  Solve[T1[z]==0,z]
```

It is convenient to draw the cut line along the negative part of the real axis, in order not to care about the negative solution of equation

$$1+3z^2+4qz^3=0 \tag{20}$$

But the two other branch points correspond to the complex solutions of equation (20). I denote the solution z with positive imaginary part with

$$z=\mathrm{NemBra(q)} \tag{21}$$

another solution is $z=\mathrm{NemBra}(q)^*$. Then, the branch point z of the inverse of the Nemtsov function can be expressed as follows:

$$z=\mathrm{NemBran(q)}=\mathrm{Nem_q}\big(\mathrm{NemBra(q)}\big) \tag{22}$$

Mathematica software happened to be slow to evaluate the "exact solution" (and even slower to plot the complex maps with it). So, the functions, and in particular, function NemBra, are implemented in C++. The implementation of function NemBra is shown in Table 1.

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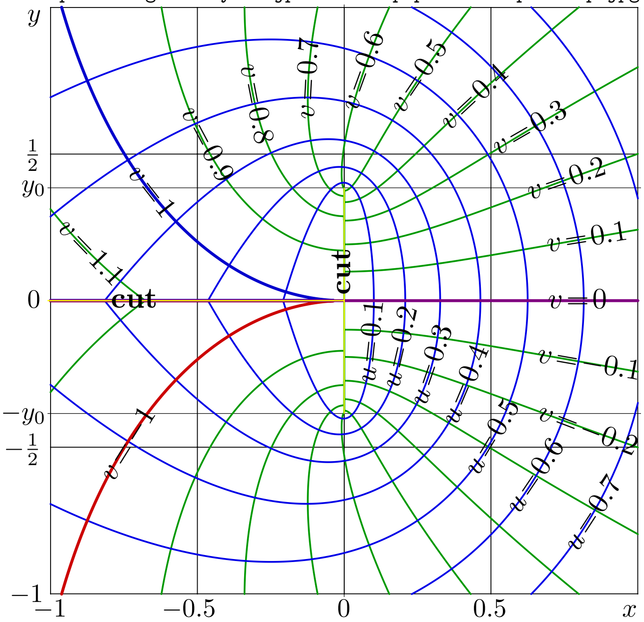

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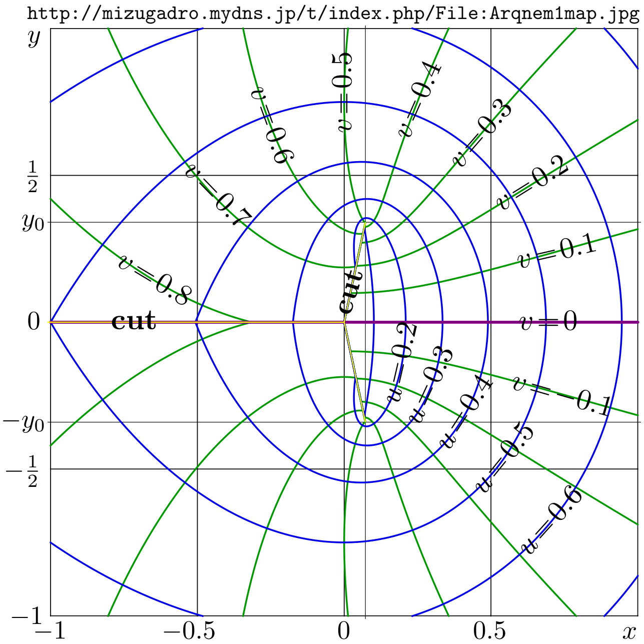

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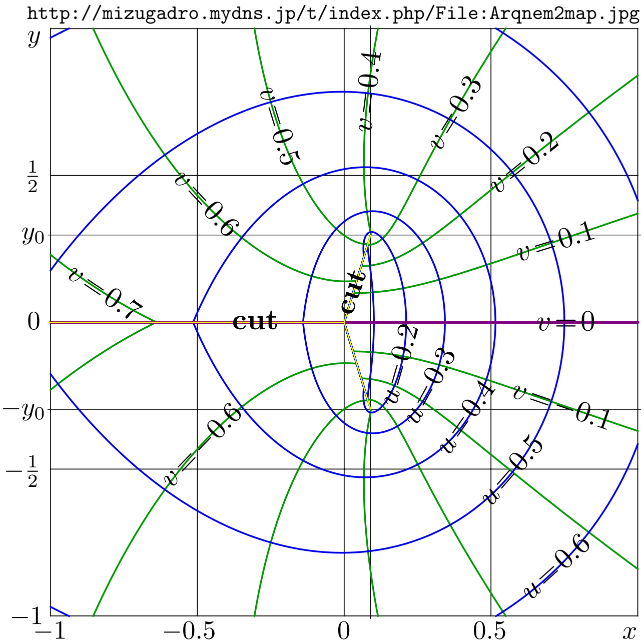

Figure 5. Maps $u+\mathrm{i}v=\mathrm{ArqNem}_q(x+\mathrm{i}y)$ for $q=0,1,2$

Table 1. C++ implementation of function NemBra

```
#define DB double
typedef std::complex<double> z_type;
#define I z_type(0.,1.)
z_type nemq(DB q,z_type z) {
   return z*(1.+z*z*(1.+q*z));}

z_type nembra0(DB q){
return 0.5773502691896258*I+
        q*(2./9+
        q*(-0.2138334330331947*I+
        q*(-0.2633744855967078 +
        q*(0.3658927631901332*I+
        q*(0.5462581923487273 +
        q*(-0.8556857213229570*I+
        q*(-1.387322393266609
        ))))))) ;}
z_type nembrao(DB q){ z_type x,y,z,s;
x=pow(z_type(-.25/q,0.),1./3.); y=x*x; z=y*y;
s=1.+y*(1.+y*(1.+y*(2./3.+z*(-2./3.+y*(-7./9.+
          z*(11./9.+y*(130./91.) )  ) ) ) ) );
return s*x;}

z_type nembra(DB q){
if(fabs(q)<.021) return nembra0(q);
if(fabs(q) >20.) return nembrao(q);
z_type Q,v,V;
Q=q*q; v=-1.-8.*Q+4.*sqrt(Q+4.*Q*Q);
V=conj(pow(v,1./3.));
return (.25/q)*(-1.+1./V+V); }

z_type nembran(DB q){ return nemq(q,nembra(q));}
```

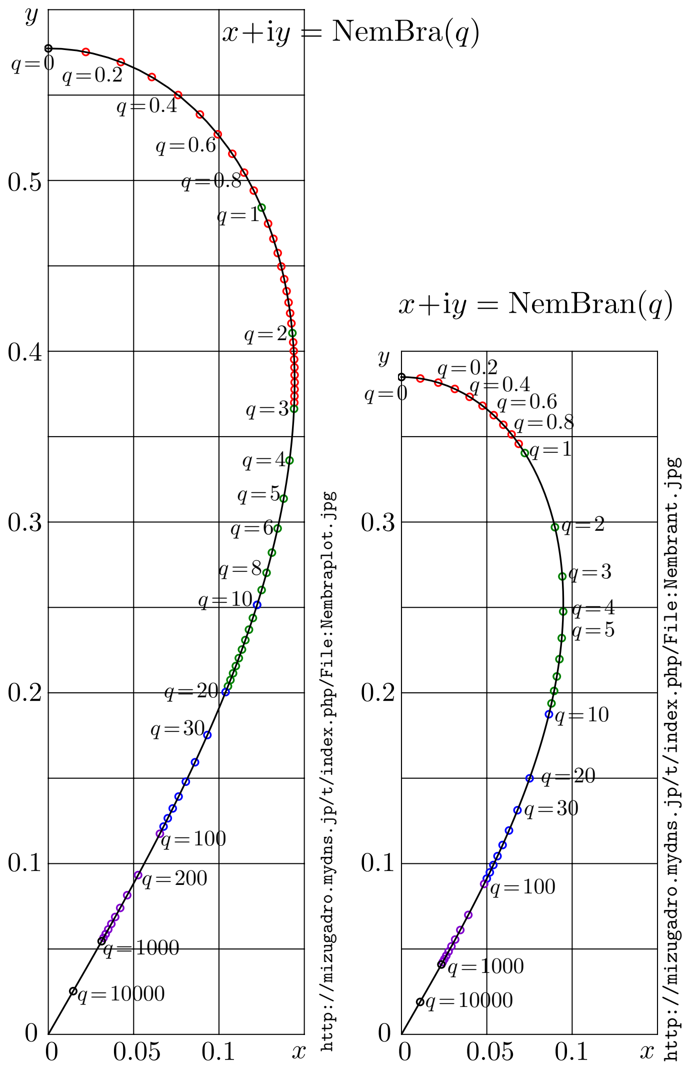

Figure 6. $x+\mathrm{i}y=\mathrm{NemBra}(q)$ and $x+\mathrm{i}y=\mathrm{NemBran}(q)$

For small values of q, and also for the big values, the asymptotic expansion are used in the implementation above, in order to keep the good precision of the evaluation. This implementation is used in generators the parametric plots of functions NemBra and NemBran in figure 6.

The same scale is used for parametric plots of functions NemBra and NemBbran in Figure 6. At small large values of parameter $q \gg 1$, (at vicinity of zero), the positions of curves these plot look similar (although, occur at pretty different values of q). At small values of z, NemBra has values, roughly, 1.5 times larger, than NemBran.

Once, the positions $B = \mathrm{NemBran}(q)$ and B^* of the branch points of function ArcNem are established, the position of cut lines can be chosen. Drawing the cut lines, it is easier to "dance" from zero, id est, to begin at the origin of coordinates. I set the cut lines as straight segments from zero to B, from zero to B^* and from zero to $-\infty$. These cut lines are marked in the maps of function ArqNem in figure 5.

The C++ implementation used In order to plot figure 5, is shown in Table 2. The algorithm is constructed from the four solutions with different positions the cut lines of each of the square roots, used in the analytic representation. With such a choice, each of the cut lines is placed at its position mentioned above.

The main goal of the implementation of function ArgNem is plotting of graphics and maps of the abelfunction of the Nemtsov function and its iterates of function at fixed values of q; this parameter and related quantities are declared as global variables in the code shown in Table 2.

Testing of the implementation by Table 2 with the Nemtsov function Nem in the complex plane confirms, that this code returns of order of 14 decimal digits. That is close to the best precision, achievable with the complex double variables. In order to keep the code easy for the human reading, the similar pieces are repeated in Table 2. However, the colleagues are invited to think, how to make the code shorter. Some other realisations of inverse o f the Nemtsov function are discussed in section 8.1.

The code by table 2 is used to plot map in figure 5, and then, for evaluation of the Abelfunction AuNem, described below, and the iterates of the Nemtsov function.

However, for the iterates, we need to construct also the supedfunction SuNem. This construction is described in the next section.

Table 2. C++ implementation of function ArqNem

```
/*  DB Q=1.; // WARNING: , tr, ti are global!
z_type nem(z_type z){ return z*(1.+z*z*(1.+z*Q)); }
z_type nem1(z_type z){ return 1.+z*z*(3.+z*(4.*Q)); }
#include"nembran.cin"
z_type  NemZo=nembra(Q);
z_type  ANemZo=nem(NemZo);
DB tr=Re(ANemZo);
DB ti=Im(ANemZo);
*/
z_type arnemU(z_type z){DB q=Q; DB q2=q*q; DB q3=q2*q;
z_type a=q-z; z_type b=1.+4.*q*z;
z_type r=81.*(a*a)+12.*(b*b*b);
z_type R=-I*sqrt(-r);
z_type s=27.*a + 3.*R; z_type S=pow(s,1./3.);
z_type B=(0.26456684199469993*S)/q
   - (1.2599210498948732*b)/(q*S);
z_type h=0.25/q2 + B;
z_type H=I*sqrt(-h);
z_type g=0.5/q2  - B + (.25+2.*q2)/(q3*H);
z_type G=I*sqrt(-g);
return - 0.25/q - 0.5*H + 0.5*G ;}

z_type arnemD(z_type z){DB q=Q; DB q2=q*q; DB q3=q2*q;
z_type a=q-z;
z_type b=1.+4.*q*z; z_type r=81.*(a*a)+12.*(b*b*b);
z_type R=I*sqrt(-r);
z_type s=27.*a + 3.*R;  z_type S=pow(s,1./3.);
z_type B=(0.26456684199469993*S)/q
  - (1.2599210498948732*b)/(q*S);
z_type h=0.25/q2 + B;
z_type H=-I*sqrt(-h);
z_type g=0.5/q2  - B + (.25+2.*q2)/(q3*H);
z_type G=-I*sqrt(-g);
return - 0.25/q - 0.5*H + 0.5*G ;}

z_type arnemR(z_type z){DB q=Q; DB q2=q*q; DB q3=q2*q;
z_type a=q-z;
z_type b=1.+4.*q*z; z_type r=81.*(a*a)+12.*(b*b*b);
z_type R=sqrt(r); z_type s=27.*a + 3.*R;
z_type S=pow(s,1./3.);
z_type B=(0.26456684199469993*S)/q
  - (1.2599210498948732*b)/(q*S);
z_type h=0.25/q2 + B;
z_type H=sqrt(h);
z_type g=0.5/q2  - B + (.25+2.*q2)/(q3*H);
z_type G=sqrt(g);
return - 0.25/q - 0.5*H + 0.5*G ;}

z_type arnemL(z_type z){DB q=Q; DB q2=q*q; DB q3=q2*q;
z_type a=q-z;
z_type b=1.+4.*q*z; z_type r=81.*(a*a)+12.*(b*b*b);
z_type R=-sqrt(r);
z_type s=27.*a + 3.*R; z_type S=pow(s,1./3.);
z_type B=(0.26456684199469993*S)/q
  - (1.2599210498948732*b)/(q*S);
z_type h=0.25/q2 + B;
z_type H=sqrt(h);
z_type g=0.5/q2  - B + (.25+2.*q2)/(q3*H);
z_type G=sqrt(g);
return - 0.25/q - 0.5*H + 0.5*G ;}

z_type arqnem(z_type z){ DB x,y; x=Re(z);y=Im(z);
if( y>ti || (x<0 && y>=0)) return arnemU(z);
//if(y<0) return conj(arnemU(conj(z)));
if(y<-ti || (x<0 && y<=0)) return arnemD(z);
if(x*ti>fabs(y)*tr) return arnemR(z);
return arnemL(z);
}
```

5 Superfunction

For the Nemtsov function Nem_q, the superfunction is solution F of the transfer equation

$$F(z+1) = \mathrm{Nem}_q(F(z)) \tag{23}$$

In analogy with approach of the previous publication [15], I look for solution F with certain asymptotic behaviour,

$$F_q(z) = \frac{1}{\sqrt{-2z}}\left(1 - \frac{q}{\sqrt{-2z}} + O\big(\ln(-z)/z\big)\right) \tag{24}$$

In order to construct the computationally-efficient asymptotic approximation of superfunction F_q, I define set of polynomials

$$P_m(z) = \sum_{n=0}^{\mathrm{IntegerPart}[m/2]} a_{m,n} z^n \tag{25}$$

where a are constant coefficients. Then, I set

$$F_{q,M}(z) = \varepsilon\left(1 - q\varepsilon + \sum_{m=2}^{M} P(\ln(-z))\varepsilon^m\right) \tag{26}$$

where

$$\varepsilon = \frac{1}{\sqrt{-2z}} \tag{27}$$

The original (and non-trivial) part of this research was to guess the representation (25),(26),(27). The following asymptotic analysis is straightforward. This analysis can be performed with the Mathematica code, see Table 3.

The expansion above can be guessed, substituting the asymptotic (24) into the transfer equation (23); the residual gives the hint, indicates the form of the next term of the expansion (26). Then I substitute the approximation (26) into the transfer equation (23). The asymptotic analysis of the residual (id est, its asymptotic minimisation) determines coefficients a.

In order to keep the code simple, the mathematica loop with m is not shown in the table.

Tens of coefficients a in equation (25) can be computed in such a way. The first coefficients are:

$$a_{2,0} = 0 \ , \qquad a_{2,1} = \tfrac{1}{4}(3 + 2q^2)$$
$$a_{3,0} = q + 3q^3 \ , \qquad a_{3,1} = -(3q)/2 - q^3$$
$$a_{4,0} = \tfrac{1}{8}(5 - 4q^2 - 44q^4), \quad a_{4,1} = \tfrac{1}{8}(-9 - 12q^2 - 4q^4),$$
$$a_{4,2} = \tfrac{3}{32}(9 + 12q^2 + 4q^4)$$
$$a_{5,0} = \tfrac{1}{12}(-39q - 104q^3 - 4q^5), \quad a_{5,1} = \tfrac{7}{4}(3q + 8q^3 + 4q^5),$$
$$a_{5,2} = -\tfrac{9q}{4} - 3q^3 - q^5$$

Assume given number M of terms in the asymptotic expansion (26). Then, the superfunction F_q can be defined as limit:

$$F_q(z) = \lim_{n\to\infty} \mathrm{Nem}_q^{\,n}\Big(F_{q,M}(z-n)\Big) \tag{28}$$

I remind, the upper index after the name of the function indicates the number of its iterate. Due to the asymptotic behaviour of $F_{q,M}$ at large negative values of the real part of argument, the limit does not depend on the number M. However, for large M, the limit converges faster.

Table 3. Computation of coefficients a in equation (25)

```
T[z_] = z + z^3 + q z^4
P[m_, L_] := Sum[a[m, n] L^n, {n,0,IntegerPart[m/2]}]
A[1, 0] = -q; A[1, 1] = 0;
a[2, 0] = 0; A[2, 0] = 0;
F[m_,z_]:=1/(-2z)^(1/2)(1-q/(-2z)^(1/2)+
Sum[P[n,Log[-z]]/(-2z)^(n/2),{n,2,m}])

m = 2;
s[m]=Numerator[Normal[Series[
 (T[F[m,-1/x^2]]-F[m,-1/x^2+1])2^((m+1)/2)/x^(m+2),
   {x,0,1}]]]
t[m] = Numerator[Coefficient[Normal[s[m]], x] ]
sub[m] = Extract[Solve[t[m] == 0, a[m, 1]], 1]
SUB[m] = Simplify[sub[m]]
f[m, z_] = ReplaceAll[F[m, z], SUB[m]]

m = 3
s[m] =Simplify[ReplaceAll[Series[
 (T[F[m,-1/x^2]]-F[m,-1/x^2+1])2^((m+3)/2)/x^(m+3),
    {x,0,0}],SUB[m-1]]];
t[m] = ReplaceAll[Normal[s[m]], Log[x] -> L];
u[m] = Table[Coefficient[t[m] L, L^n]==0,
 {n,1,1+IntegerPart[m/2]}];
tab[m] = Table[a[m, n], {n, 0, IntegerPart[m/2]}];
sub[m] = Extract[Solve[u[m], tab[m]], 1]
SUB[m] = Join[SUB[m-1], sub[m]];

(* and so on for  m=4,  m=5, etc. *)
```

The asymptotic representation (25),(26),(27) approximates the superfunction F at least for large negative values of the real part of the argument. For other values of argument, the transfer equation is used so many times, as necessary to bring the argument into the range, where the asymptotic approximation provides the required precision.

The numerical evaluation of superfunction described above can be programmed in any language, even at Mathematica. However, the mathematica representation is a little bit slow for the evaluation (to press a key, to have a tea). So, this function is implemented in C++; the code can be extracted from the generators of the pictures.

Explicit plot of function F_q, constructed in such a way, is shown in figure 7.

At large M, the limit in (28) converges quickly. For q of order of unity, and argument z of order of unity, with $M\!=\!30$, it is sufficient to make $n\!=\!20$ iterates in order to approximate limit in equation (28) with 14 decimal digits (that is close to limit for the complex double variables).

At large negative value of x, function $F_q(x)$ decreases with increase of q; this is determined by the first term, dependent on q, in the expansion 26. However, the next term increase with increase of q, so, the curves in figure 7 cross each other. The scales of the abscissas and the ordinates in figure 7 differ for 100 times, in order to make the crossing of the curves visible.

The transfer equation has translational invariance. If some $z \to F(z)$ is the solution, then, for a constant C, function $z \to F(z\!+\!C)$ is also solution, id est, also superfiunction of the same transfer function.

In order to make figures more beautiful, it is convenient, that at zero, the superfunction has value unity. For this reason, I define superfunction SuNem_q as superfunction

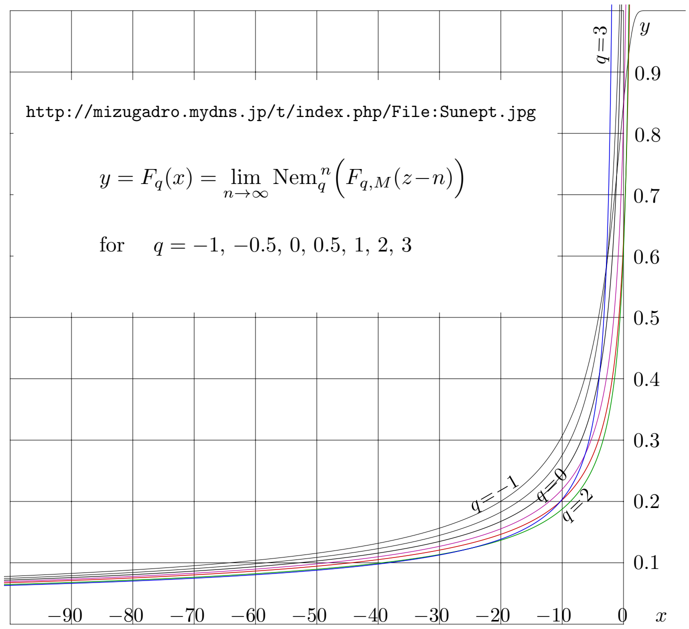

Figure 7. $y\!=\!F_q(x)$ by (28) for various q

F with displaced argument:

$$\mathrm{SuNem}_q(z) = F_q(x_1\!+\!z) \qquad (29)$$

where $x_1 = x_1(q)$ is real solution of equation

$$F_q(x_1) = 1 \qquad (30)$$

Dependence $x_1(q)$ is shown in figure 8. The dependence is not monotonous; in order to show better the general trend, the curve is a little bit extended to the negative values of q.

I

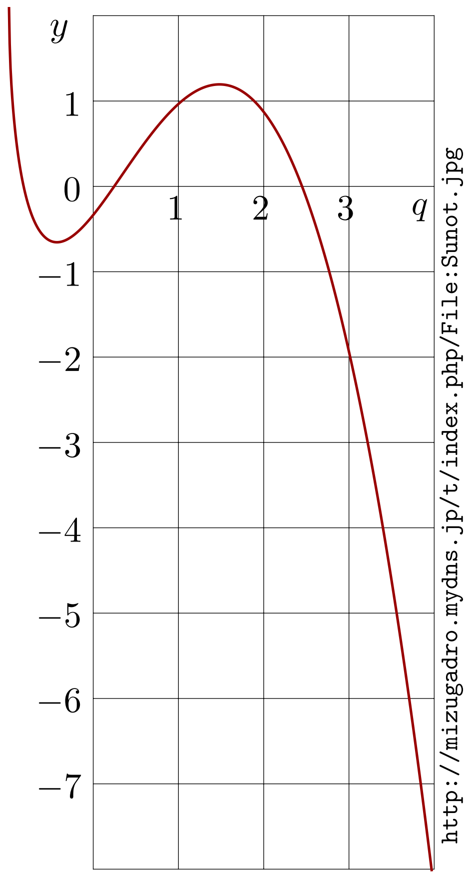

Figure 8. $y\!=\!x_1(q)$

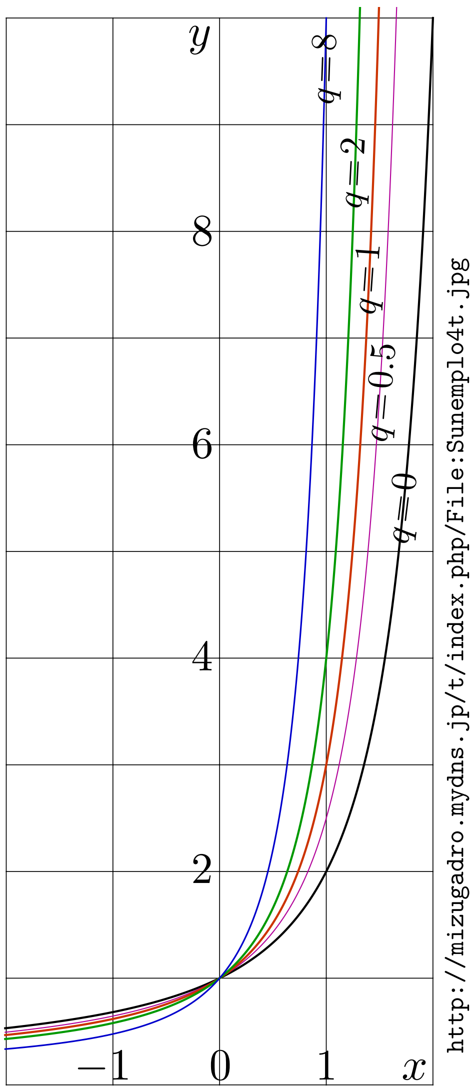

Figure 9. $y = \mathrm{SuNem}_q(x)$ for various q

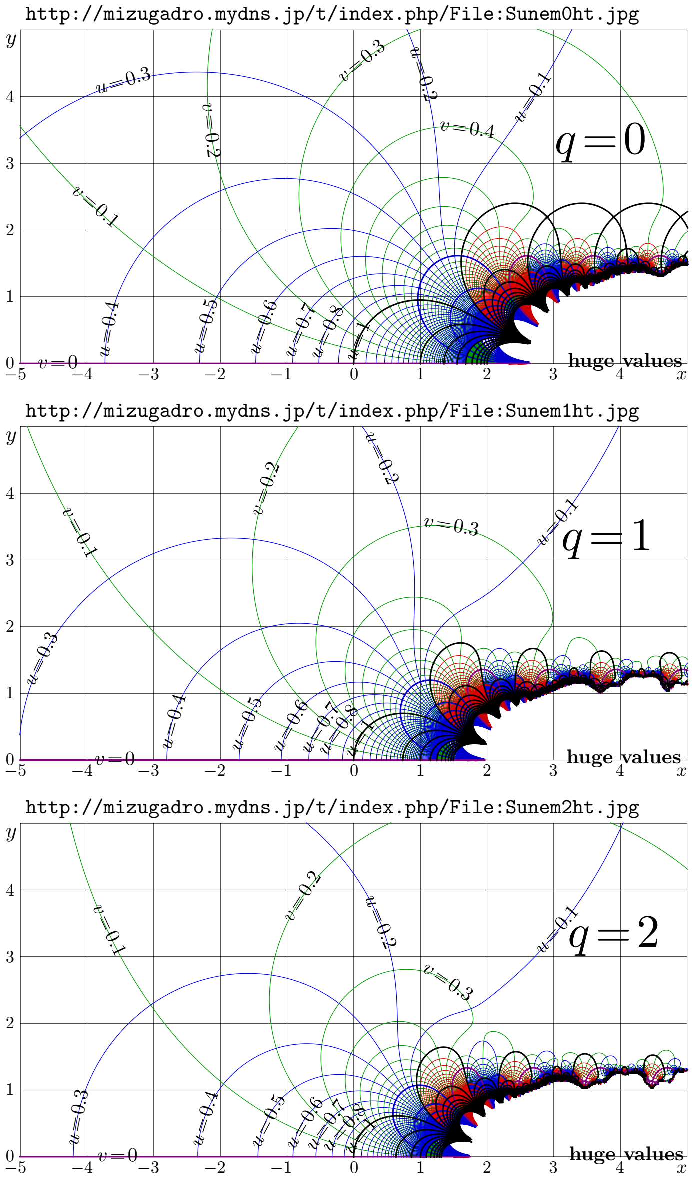

Figure 10. $u+\mathrm{i}v = \mathrm{SuNem}_q(x+\mathrm{i}y)$, for $q=0,1,2$

For real values of x, function $\mathrm{SuNem}_q(x)$ is shown versus x in figure 9. It is analogy of function F_q shown in figure 7; the only, the curves are displaced; this displacement depends on q. This displacement as funtion of q is shown figure 8.

Superfunction SuNem_q shown in figure 9 is easier to interpret, than the "not displaced" original superfunction F_q shown in figure 7: there is only one point, where the curves for different q intersect. The larger is q, the steeper is the superfunction. That corresponds to the native expectation. The special choice $\mathrm{SuNem}_q(0) = 1$ leads to certain advantages: at integer q, at integer values of the argument, the function takes integer values. This helps to trace and correct errors (if any) in the implementation.

Complex maps of superfunction SuNem_q are shown in figure 10 for $q = 0$, $q = 1$, and for $q = 2$. These maps are symmetric with respect to reflection from the real axis, so, the only upper half plane is shown.

The superfunction SuNem approaches the fixed point 0 of the Nemtsov function in namy directions: at large negative values of the real part of the argument, and at large values of the imaginary part. The only in the half-strip of widrh of order 3 allong the positive part of the real axis, it shows fast growth and fast oscillations.

Function SuNem_q by (26),(28),(29)(30) seems to be the only solution of the transfer equation (23) with additional condition $\mathrm{SuNem}_q(0) = 1$, that approaches 0 at $-\infty$ and at $\pm\mathrm{i}\infty$. This can be formulated as conjecture:

Conjecture 1. There exist real-holomorphic solution $F = \mathrm{SuNem}_q$ of equation (23) that satisfies the additional condition $\mathrm{SuNem}_q(0) = 1$ and has the following asymptotical properties:

$$\lim_{x\to-\infty} \mathrm{SuNem}_q(x+\mathrm{i}y) = 0 \text{ for any real } y \tag{31}$$

$$\lim_{y\to-\infty} \mathrm{SuNem}_q(x+\mathrm{i}y) = 0 \text{ for any real } x \tag{32}$$

There is only one such a solution.

Other solutions (showing growth along some lines toward $\mathrm{i}\infty$ and $-\mathrm{i}\infty$ can be constructed as follows:

$$f(z) \tag{33}$$

Any holomorphic function, periodic along the real axis, shows fast (at least exponential) growth in some directions toward the imaginary axis. So, such a modified function should not satisfy criterion

For computation of iterates of the Nemtsov function Nem_q, the superfunction SuNem_q is necessary, but not sufficient. Also, the Abelfunction is required. The Abelfunction is described in the next section.

6 Abelfunction

I search for the abelfunction $G_q = F_q^{-1}$, In the following form:

$$\mathrm{G}_{q,M}(z) = -\frac{1}{2z^2} + \frac{q}{z} + \frac{2q^2+3}{2}\log(z) + \sum_{n=0}^{M} c_n z^n \qquad (34)$$

$$G_q(z) = \mathrm{G}_{q,M}(z) + O(z^{M+1}) \qquad (35)$$

Coefficients c depend on q. These coefficients can be computed either with asymptotic analysis of equation

$$G_{q,M}(F_{q,M}(x)) = z \qquad (36)$$

or from the Abel equation

$$G_q(\mathrm{Nem}_q(z)) = G_q(z) + 1 \qquad (37)$$

It is easy to calculate

$$c_0 = \frac{q^2}{2} + \frac{1}{4}\left(2q^2+3\right)\log(2) \qquad (38)$$

it follows from $G_q(F_q(z)) \approx z$ at large values of $-z$.

Few more coefficient can be calculated from the asymptotic analysis "with paper and pen"; even more coefficients can be calculated in Mathematica with code in Table 4.

Table 4. Calculation of coefficients c in equation (34)

```
T[z_]=z+z^3+q z^4
P[m_,L_]:= Sum[a[m, n] L^n, {n, 0, IntegerPart[m/2]}]
F[m_,z_]:=1/(-2z)^(1/2)(1-q/(-2z)^(1/2)+
          Sum[P[n,Log[-z]]/(-2z)^(n/2),{n,2,m}])
G[m_,x_]:=-1/(2x^2)+q/x+q^2/2+1/4(3+2q^2)Log[2]+
             1/2 (3+2q^2)Log[x]+Sum[c[n]x^n,{n,1,m}]
Series[ReplaceAll[F[3,h+G[3,z]],
     a[2,1]->1/4 (3+2q^2)],{z,0,4}]
m=1;
sg[m]=Coefficient[Series[G[m+3,T[z]]-G[m+3,z]-1,
                {z,0,3}],z^(m+2)]
st[m]=Solve[sg[m]==0,c[m]]
su[m]=Extract[st[m],1]
SU[m]= su[m];
m=2;
sf[m]=Series[ReplaceAll[G[m+3,T[z]]-G[m+3,z]-1,
           SU[m-1]],{z,0,m+2}]
sg[m]=Simplify[Coefficient[sf[m] 2^m,z^(m+2)]]
st[m]=Solve[sg[m]==0,c[m]]
SU[m]=Join[SU[m-1],su[m]]
m=3;
sf[m]=Series[ReplaceAll[G[m+3,T[z]]-G[m+3,z]-1,
            SU[m-1]],{z,0,m+2}]
sg[m]=Simplify[Coefficient[sf[m] 2^m,z^(m+2)]]
st[m]=Solve[sg[m]==0,c[m]]
su[m]=Extract[st[m],1]
SU[m]=Join[SU[m-1],su[m]]
(*and so on for m=4, m-5, etc... *)
```

From the asymptotic expansion, abelfunction G_q can be defined as limit

$$\mathrm{G}_q(z) = \lim_{n\to\infty}\Big(\mathrm{G}_{q,M}\big(\mathrm{ArqNem}_q^{\ n}(z)\big) + n\Big) \qquad (39)$$

Due to the asymptotic behaviour of $F_{q,M}$, the limit does not depend on number M of terms kept in the expansion; however, the larger M, the faster is conversion of the limit. Then, function $\mathrm{AuNem_a}$ can be expressed as

$$\mathrm{AuNem}_q(z) = G_q(z) - G_q(1) \qquad (40)$$

The explicit plot is shown in figure 11 for 3 values of q. Complex maps of function AuNem_q are shown in figure 12 for the same values of parameter, $q=0$, $q=1$, $q=2$.

As in the case of maps of SuNem, maps of AuNem have symmetry (common for all real-holomorphic functions), so, the only upper half of the complex plane is shown.

Evaluation of function AuNem_q begins with evaluation of function ArqNem_q, and AuNem_q has the same brach points and the same cut lines, as function ArqNem_q.

The precision of implementation used for functions AuNem_q can be verified, substituting it into the Abelequation; relation $\mathrm{AuNem}_q(\mathrm{Nem}_q(z)) = \mathrm{AuNem}_q(z)+1$ is reproduced with 14 decimal digits in the most of the complex plane (except the branch points and the cut lines). Similar test can be performed with relation $\mathrm{SuNem}_q(\mathrm{AuNem}_q(z)) = z$ as both functions, SuNem_q and AuNem_q are already implemented.

The readers are invited also to check, how important is choice of appropriate branch of the inverse the Transfer function, trying to plot the map with another placement of the cut lines, inverting the Nemtsov function. Choice of the cut lines of the inverse function is important for iterates of any holomorphic function.

Once both, the superfunction SuNem_q and the abelfunction AuNem_q for the Nemtsov function are constructed, its iterates can be expressed. These iterates are described in the following section.

7 Iterates

With functions SuNem_q and SuNem_q from previous sections, iterates of the Nemtsov function can be defined with (4); for $F = \mathrm{SuNem}_q$ and $G = \mathrm{AuNem}_q$, I re-write it as follows:

$$\mathrm{Nem}_q^{\ n}(z) = \mathrm{SuNem}_q\Big(n + \mathrm{AuNem}_q(z)\Big) \qquad (41)$$

As usually, the number n of iterates here has no meed to be integer. It can be real, and even complex. Dependence of the iterate on the number n, at fixed argument

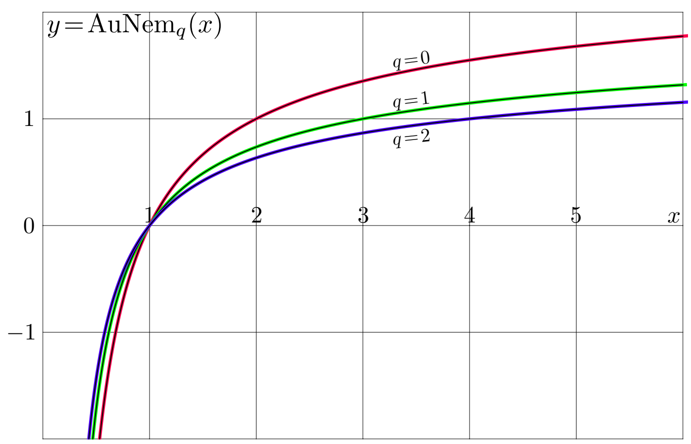

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Figure 11. $y = \mathrm{AuNem}_q(x)$ for $q=0$, $q=1$, $q=2$.

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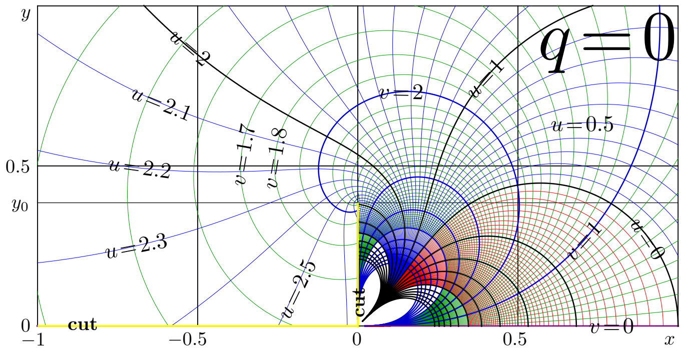

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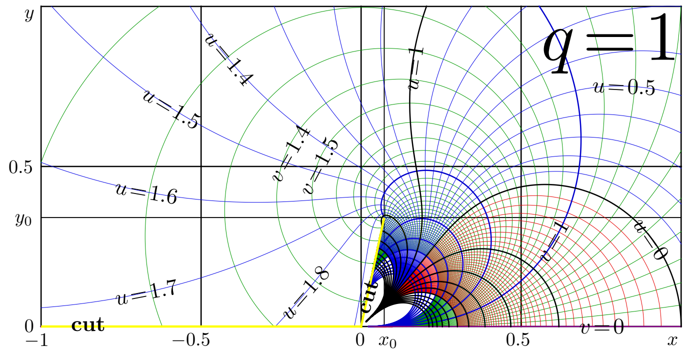

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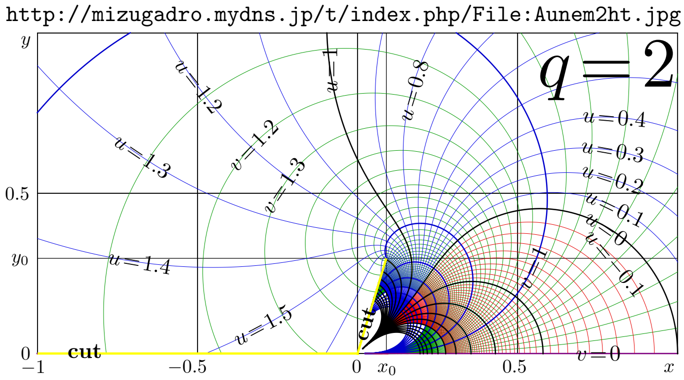

Figure 12. $u+\mathrm{i}v = \mathrm{AuNem}_q(x+\mathrm{i}y)$ for $q=0$, $q=1$, $q=2$

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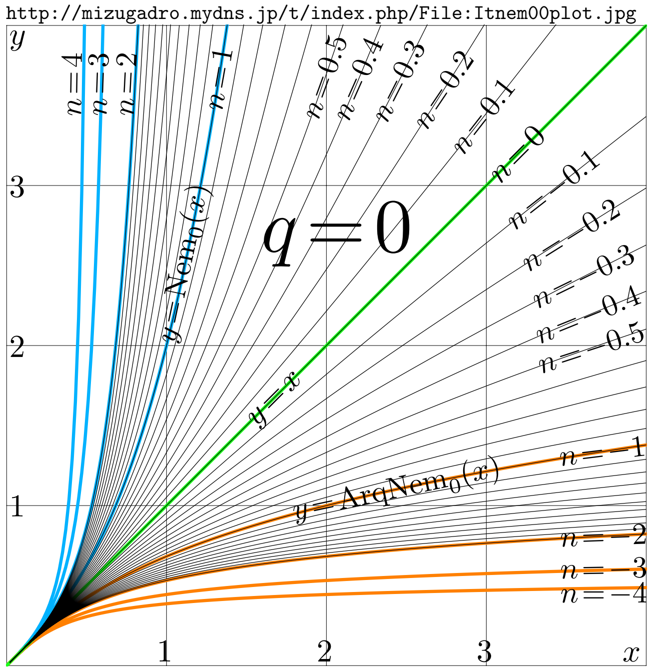

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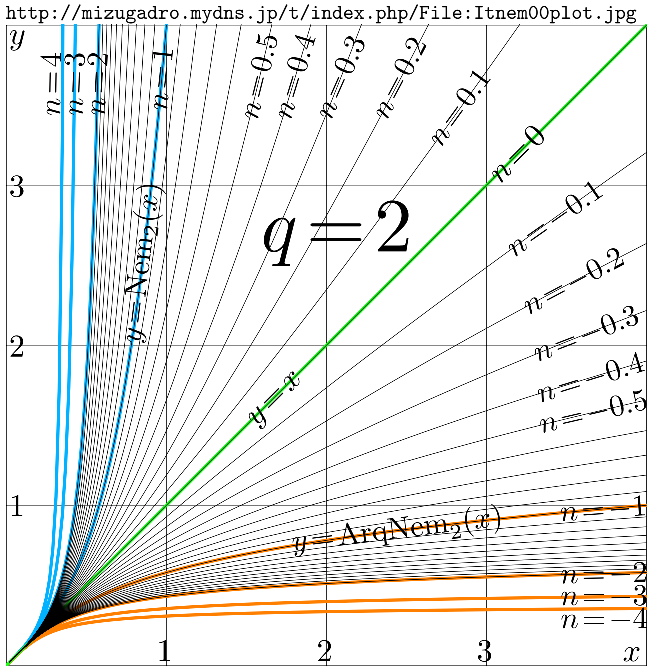

Figure 13. $y = \mathrm{Nem}_q(x)$ for $q=0$ and for $q=2$ for various n

z, appears just a superfunciotn SuNem)q with argument, displaced with a constant z.

Explicit plots of the iterates for various n are shown in figure 13. This is analogy of plots of iterates of the power function, shown in figure 3. At large values of the argument, the curves look similar to those for the power function, as it was expected. At moderate values, the difference is seen. In particular, the Nemtsov function has only one real fixed point, where the curves of the iterates touch each other.

Iterates shown in figure 13 look similar to iterates of other quickly growing holomorphic functions considered previously [6, 7, 8, 10, 13, 16]. In particular, at $n\approx 0$, the iterate Nem_q^n looks similar to identical function; at $n=1$, it is just Nemtsov function Nem_q, and at $n=-1$, it is the inverse function, id est, ArqNem_q.

For parameter $q=1$, the complex maps of the iterates are shown in figure 15 with lines $\Re(\mathrm{Nem}_1(x+\mathrm{i}y) = \mathrm{const}$ and lines $\Im(\mathrm{Nem}_1(x+\mathrm{i}y) = \mathrm{const}$. As in the explicit plots, at large values of argument, the maps look similar to maps iterates of the power function, see figure 4. At moderate values of the argument, say, smaller than unity, the qualitative difference is seen. In particular, the complex branch points and the corresponding cut lines are seen. In order to show them better, the part with $-1<x<1$, $0<y<1$ is shown.

Iterates of a growing real-holomorphic function are also real-holomorphic; the complex maps are symmetric with respect to reflection from the real axis, so, the only upper half of the complex plane is shown shown in figure 14. At large values of the argument, the maps look similar to maps for the power function; one example is shown in figure The left column shows maps for the positive iterates; the number n varies from 0.6 at the top map with step -0.1 to 0.1 at the bottom map. In the similar way, the right hand side column represents maps for n from -0.6 at the top to -0.1 at the bottom. Only case with $q=1$ is presented, but one can download the generator of the figure and plot similar maps for other values of q, and, of course, other values of number n of iterate; this number can be even complex.

For better comparison of the maps of iterates of the

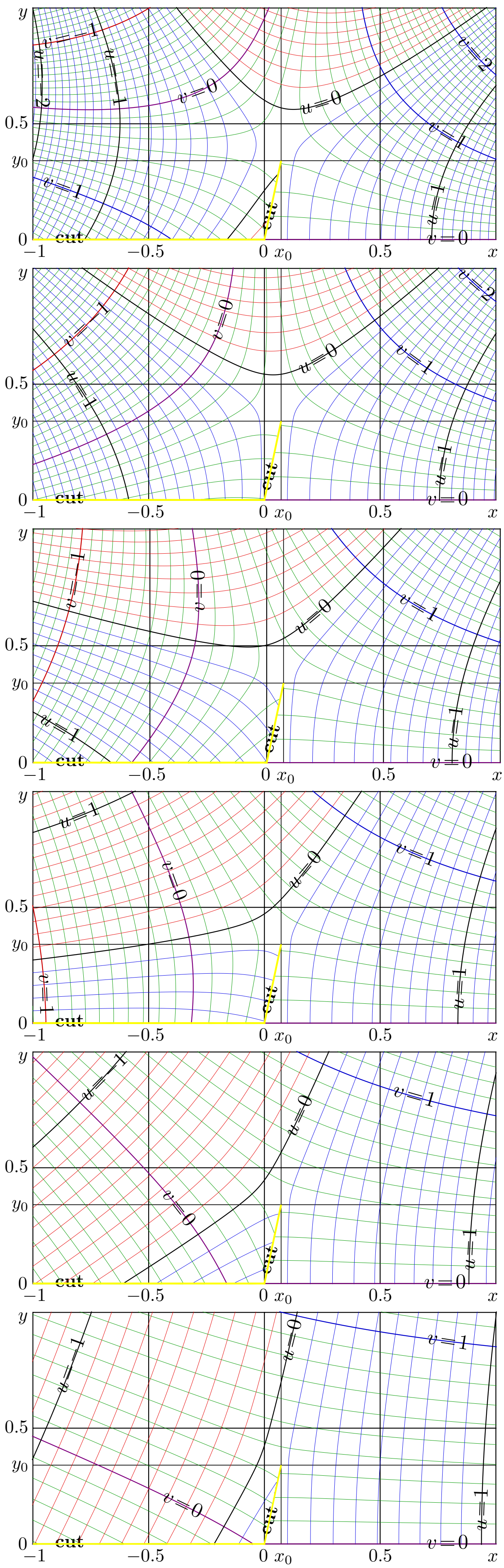

Figure 14. $u+\mathrm{i}v=\mathrm{Nem}_1^n(x+\mathrm{i}y)$ by (41) for $n=0.6\ ..\ 0.1$

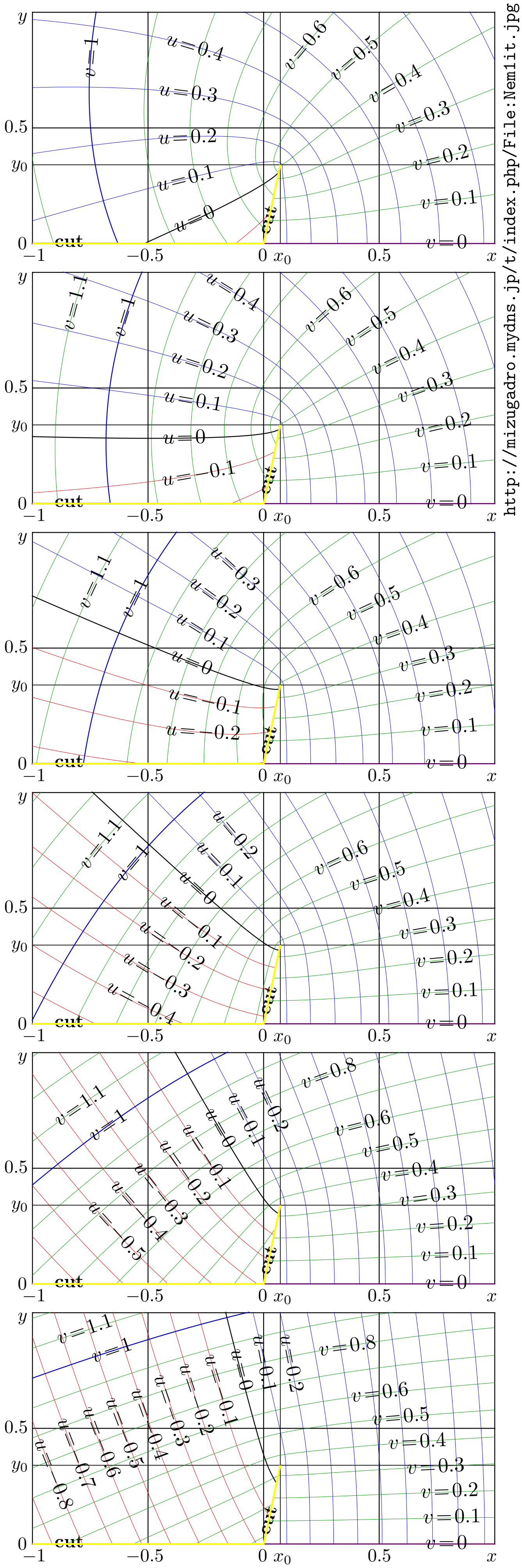

Figure 15. $u+\mathrm{i}v=\mathrm{Nem}_1^n(x+\mathrm{i}y)$ by (41) for $n=-0.6\ ..\ -0.1$

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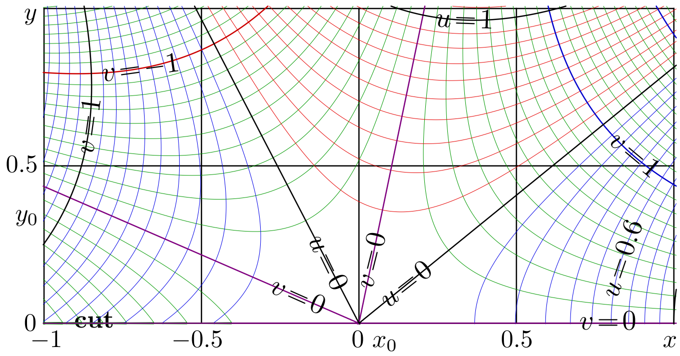

Figure 16. $u+\mathrm{i}v=(x+\mathrm{i}y)^{2.3}$

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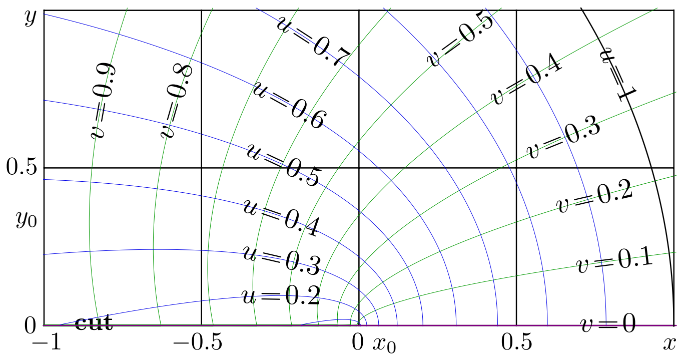

Figure 17. $u+\mathrm{i}v=(x+\mathrm{i}y)^{1/2.3}$

Nemtsov function to those of the power function, in figures 16, 17, the maps of the power functions $z\mapsto z^s$ and $z\mapsto z^{-s}$ are shown in the same scale and in the same range, as in the top pictures of figures 14, 15. The power parameter $s\!=\!2.3$ is chosen close to the iterates of the highest term of the Nemtsov function (id est, power 4), iterated 0.6 times.

The map in figure 16 is created with lines
lines $u=\Re\big((x+\mathrm{i}y)^s\big)=\mathrm{const}$ and
lines $x=\Im\big((x+\mathrm{i}y)^s\big)=\mathrm{const}$

In the similar way,
lines $u=\Re\big((x+\mathrm{i}y)^{-s}\big)=\mathrm{const}$ and
lines $x=\Im\big((x+\mathrm{i}y)^{-s}\big)=\mathrm{const}$
are drawn in figure 17.

Semehence of map in figure 16 with the top map in figure 14, as well as map in figure 17 with the top map in figure 17 indicates, that the constructed iterates follow the preliminary and intuitive expectation about non-integer iterates of the Nemtsov function.

Iterates by (41) shown in figures 13, 14 provide the smooth (holomorphis) transition from the Nemtsov function Nem_q to the identity function and then to the inverse function ArqNem. Iterates have the group property,

$$T^{m+n}(z)=T^m(T^n(z)) \tag{42}$$

This ratio holds only for certain range of values of parameters, that includes the positive part of the real axis for z. In particular, for $n\!=\!m\!=\!1/2$, we have composition of the two half iterates of the transfer function T.

Let

$$h=\mathrm{Nem}_1^{1/2} \tag{43}$$

be half iterate of the Nemtsov function with parameter $q\!=\!1$. Then, we may expect, that, in some range, z the

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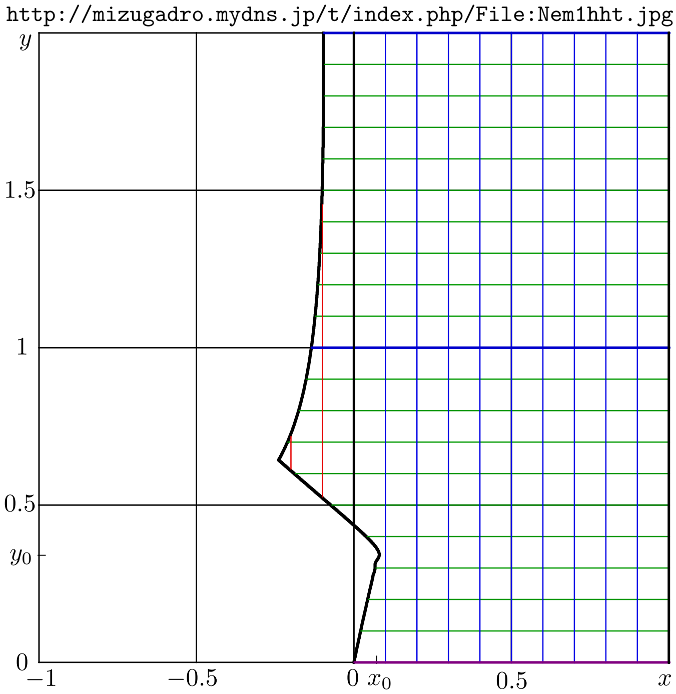

Figure 18. Range of validity of relation (44) in plane $x=\Re(z)$, $y\!=\!\Im(z)$ is shaded with rectangular grid

relation

$$h(h(z))=\mathrm{Nem}_1(z) \tag{44}$$

holds. For $z=x{+}\mathrm{i}y$, this range is shaded in figure 18. The boundary of this range is shown with tick curve. Technicaly, the picture is just map of the identity function for those values of the argument, while the equation (44) holds with at least 10 significant decimal figures.

Figure 18 illustrates the general meaning of iterate half of a real holomorphic function - in the same sense, as it is interpreted, in previous publications [1, 10, 18].

8 Discussion

This article describes the computation of the following functions:
1. Nemtsov function Nem_q (which is polynomial and can be evaluated directly from its definition),
2. The inverse function $\mathrm{ArqNem}_q\!=\!\mathrm{Nem}_q^{-1}$
3. Supervunction SuNem_q, that is original result of this publication,
4. Abelfunction $\mathrm{AuNem}_q\!=\!\mathrm{SuNem}_q^{-1}$,
5. Iterates $\mathrm{Nem}_q^{\,n}$, that follows the preliminary expectations in wide range of values of its argument (and, in particular, in some vicinity of the positive part of the real axis) for various values of parameterr q.

As usually, per each question, answered by the research described, some new questions arise, that may be subject for the future research. Some of them are mentioned below.

8.1 More inverse functions

Computation of the inverse function ArqNem happened to be a little bit tricky. (Actually, the handling took more efforts, than the upgrade the algorithm for the symmetric transfer function $T(-z)=-T(z)$ by [15] to a little bit more general case.) Function ArqNem, described in section 4, is actually the 4th attempt to construct Nem^{-1}

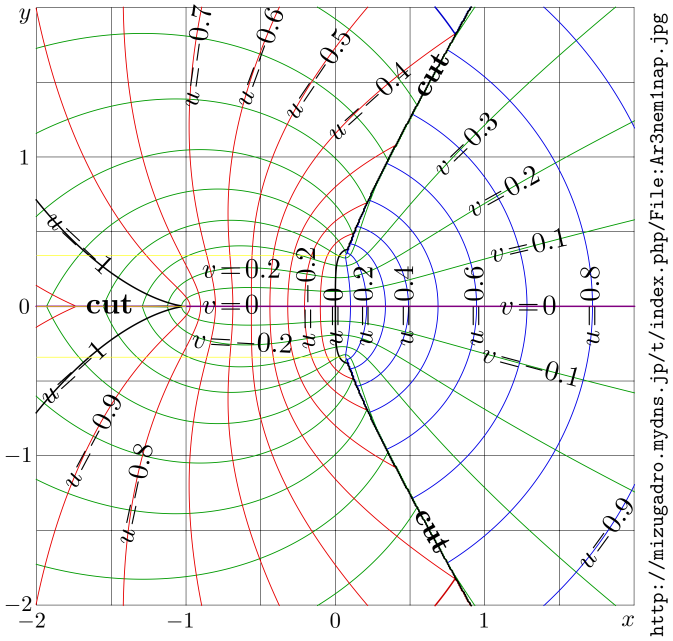

Figure 19. $u+\mathrm{i}v = \mathrm{ArnemR}(x+\mathrm{i}y)$

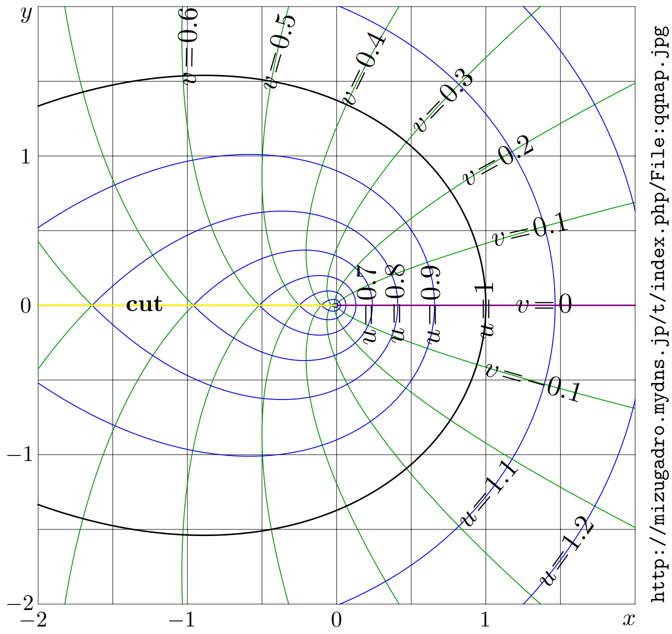

Figure 20. $u+\mathrm{i}v = (x+\mathrm{i}y)^{1/4}$

in a way, usable for the efficient numerical computation. After to see maps of the resulting abelfnction (section 6), it is easier to understand, why namely ArqNem with cut lines shown in figure 5, is chosen for construction of function AuNem. (figure 12). The behaviour is discussed in this subsection.

The naive attempt to use Mathematica routine Solve and choice of the real growing-up solution (it happen to be the the third one, among 4 solutions) leads to function denoted here with arnemR. For $q=1$, the complex map is shown in figure 19.

The implementation can be reduced to the code shown in table 5; DB means "double", and z_type means "complex<double>"; Q has sense of q and appears as global variable.

In the right hand side, the map at figure 19 looks similar to the map of the power function, say, $z \mapsto z^{1/4}$, shown in figure 20.

However, the cut line, shown in figure 19, break the similarity. This cut line would appear also at the abelfunction for the Nemtsov function; this would make difficult the analogy of iterates of the Nemtsov function with that of the power function (corresponding to the leading term in the Nemtsov function at large values of the argument.

Table 5. Code for evaluation of the naive inverse of Nem_q

```
z_type arnemR(z_type z){DB q=Q, q2=q*q, DB q3=q2*q;
z_type a=q-z; z_type b=1.+4.*q*z;
z_type r=81.*(a*a)+12.*(b*b*b);
z_type R=sqrt(r); z_type s=27.*a + 3.*R;
z_type S=pow(s,1./3.);
z_type B=(0.26456684199469993*S)/q
        - (1.2599210498948732*b)/(q*S);
z_type h=0.25/q2 + B;
z_type H=sqrt(h);
z_type g=0.5/q2 - B + (.25+2.*q2)/(q3*H);
z_type G=sqrt(g);
return - 0.25/q - 0.5*H + 0.5*G ;}
```

For this reason, other versions of Nem_q^{-1} are constructed. They are arranged from function arnemR above, playing with the expression of R through r, H through h, G through g in the code above, dependently on value of argument. This allows to choose other branches of the inverse function. Then, other functions are constructed: ArcNem, Arknem and Arqnem. For the same value $q=1$, the complex maps are shown in fugure 21. The maps show, how, gradually, the cut lines can be moved, approaching the negative part of the real axis. At this movement, the map becomes more similar to the map for the corresponding power function. It is supposed to be so: at large values of z, the only leading term of the polynomial $\mathrm{Nem}_q(z)$ is important; and the inverse function also should look similar to that of the corresponding power function.

The last of these unverse functions, namely ArqNem, has no cut lines at infinity, expect the cut along the negative part of the real axis; the same cut, as in the case of the power function with non-integer parameter. For this reason, function ArqNem is chosen to build-up the abelfunction of the Nemtsov function and the corresponding iterates.

The readers are invited to plot the abel function for the Nemtsov function, using ArnemR, ArcNem or ArkNem instead of ArqNem, and see, how the cut lines appear at the map of the abelfunction.

For the Nemtsov function, the choice of the inverse, ArqNem, can be guessed from the analogy with the power funcion. In general case, for iterates of other functions, it is difficult to guess a priori, which inverse function provides the wide range of holomorphism of the abelfunction.

tIn general case, it may be non-trivial, to guess the appropriate position of the cut line of the inverse of the transfer function such that the abelfunction has wide range of holomorphism. It may be matter for the future research,

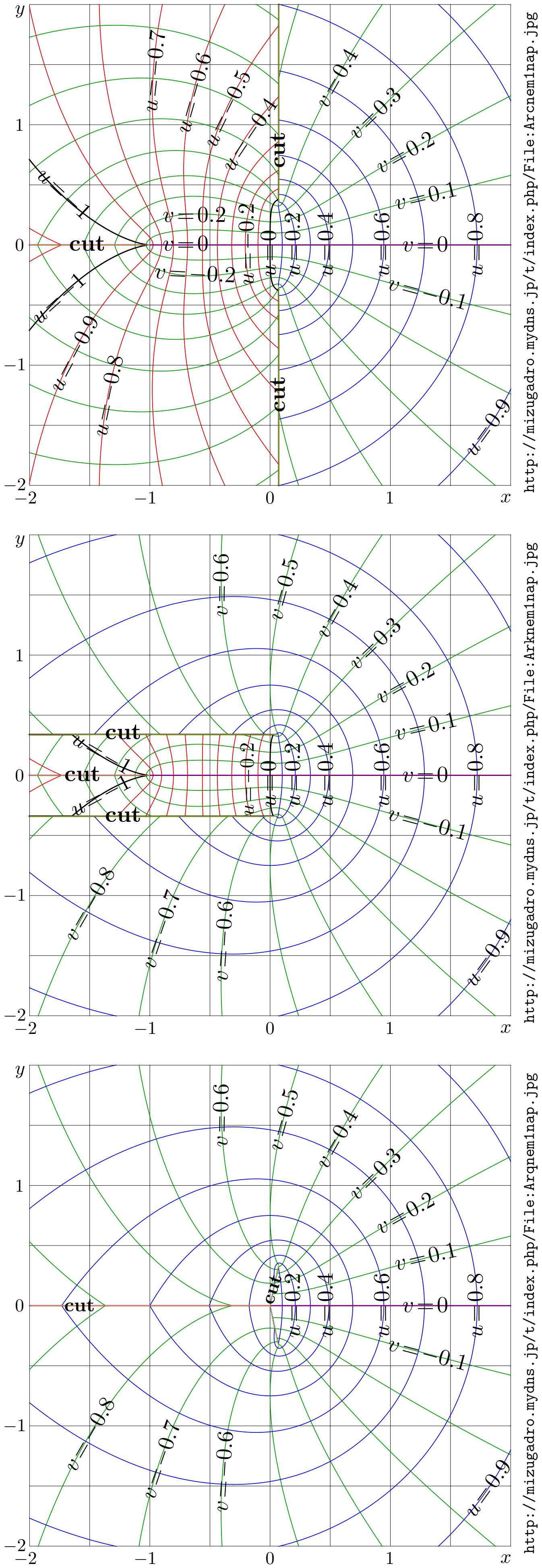

Figure 21. $u+\mathrm{i}v=f(x+\mathrm{i}y)$ for $f=\mathrm{ArcNem}_1$, $f=\mathrm{ArkNem}_1$ and $f=\mathrm{ArqNem}_1$

8.2 Uniqueness

Here, I focused mainly in the computational part. I expect, the colleges can download the C++ "complex double" implementations of all these functions and modify the suggested plots to the new cases, if they cause any doubts. The same refer to the Mathematica codes, used to generate these C++ codes. So, if any error or misprint in the formulas above, it can be revealed with numerical and/or analytical computations with the codes supplied.

Another reason, why the consideration of the functions in the complex plane, refer to uniqueness of superfunction. The view in the complex map gives a hint, what additional requirements would be applied in order to provide the uniqueness.

In general, the superfunction is not unique. The iterates of the power function, that correspond to dominant term of a polynomial at large argument, can be used as asymptotic for the superfunction, abelfunction and the iterate. The substitution of such a representation to the transfer equation gives a hint, which terms should be added to the superpower function in order to get asymptotic of the super polynomial. Then, iterates of inverse of the polynomial, allows to extend the range of validity and calculate the superfunction with any required number of significant figures. However, the resulting superfunction will have the cuts, corresponding to cuts of the inverse of the Nemtsov function used; so, the resulting superfunction is not entire.

In general, for a given transfer function, the superfunction (and, therefore, the corresponding abelfunction) is not unique. Other superfunctions f for $T=\mathrm{Nem}_q$ can be constructed with the appropriate modification of the argument of function SuNem_q:

$$f(z)=\mathrm{SuNem}_q(z+\zeta(z)) \qquad (45)$$

where $\zeta(z)$ is holomorphic periodic function with period unity. Such a periodic function must show fast (at least exponential) growth at least in some directions in the complex plane, for example, in the direction of the imaginary axis, at least for some values of the real part of the argument; or even has cut lines and/or singularities. In such a way, function $f=\mathrm{SuNem}_q$ seems to be the only solution of the transfer equation $f(z{+}1)=T(f(z))$ with additional condition $f(0)=1$, bounded at least in the left hand side of the complex half plane; say,

$$\forall\{x,y\in\mathbb{R}, x<0\},\ |f(x+\mathrm{i}y)|<1 \qquad (46)$$

The conjecture above is verified with attempts to construct such a function and the numerical analysis with many (of order of 14) decimal digits in wide area of the complex plane. However, such a numerical test cannot substitute the rigorous mathematical proof of this conjecture. Such a proof (perhaps, for the more general case) can be subject of the future research.

The Nemtsov function $\mathrm{Nem}_q(z)=z+z^3+qz^4$ is important and interesting as example of a transfer function, for which the superfunction SuNem_q and/r its abelfunction could not be calculated with methods published earlier. The efficient algorithms for the evaluation are described above and implemented: every complex map, described above, can be generated at any computer in real time with the C++ code suggested. However, following the description, the same code can be re-written in any other programming language.

Construction of the algorithm for the efficient evaluation is continuation of multiple attempts to define, to invent any real-holomorphic transfer function such that its superfunction and abelfunction cannot be constructed in a compact form, that allow the fast and precise evaluation.

8.3 Toward the general tool

This research can be considered as a part of the ambitious project of authomatic of construction of superfunctions and abelfunctions for a given holomorphic transfer function; at least, for the real-holomorphic transfer functions. The algorithm should search for the fixed points of the transfer function, and, if they exist, choose some of them as asymptotic of the superfunction (and abelcunction) at large values of the argument (for example, at i∞), and the appropriate value of the superfunction at some point (for example, $F(0)=1$).

If there exist a real fixed point L sigh that $T'(L)>0$, $T'(L)\neq 1$, then, the "regular iteration" can be used to construct the superfunction, as it is cone for the exponential to base $b<\exp(-1/\mathrm{e})$, for factorial and for the quadratic transfer function ("logistic operator") or for the tetration itself as a transfer function (to construct pentation as the next holomorphic ackermann function) [8, 10, 11, 14]. However, even in this case, some denominators in the higher terms of the asymptotic expansion of the superfunction may happen to be zero; this may limit the number of terns in the primary approximation and reduce the efficiency of the resulting evaluation of superfunction.

The fixed points may happen to be complex, as it takes place in the exponential to base $b>\exp(-1/\mathrm{e})$; then, the Cauchi integral [6] or the William Paulsen's method [18, 19] can be used for the efficient evaluation.

Derivative of the transfer function may happen to be unity, as it takes place for exponential to base $b=\exp(-1/\mathrm{e})$ [9], for sin [15] and for the Nemtsov function above. In this cases, the logarithmic terms appear in the asymptotic expansion of the superfunction, and this should be revealed.

En fin, it may happen, that the transfer function has no fixed point at all, at it takes place for the "Trappmann function" $z\mapsto x+\exp(z)$ [13]. Consideration of the example shows, that even in such a case, it is possible to guess the asymptotic of the superfunction at infinity and use it for the algorithm for the efficient evaluation of the superfundtion and abelfunction.

Realisation of this project is expected to convert construction and evaluation of superfunction to a routine, in a way, similar to calculation of derivatives and integrals of special functions in the high level programming languages. However, for the technical support of the routine mentioned, the algorithm should "know", how to deal with a special cases, and, in particular, with case $T'(L)=1$, $T''(L)=0$.

Consideration of the Nemtsov function seems to be important and unavoidable step on this analysis. Other similar functions can be treated with the same algorithm of construction of superfunction and abelfunction.

The efficient calculation of superfunctions, abelfunctions and iterated may have wide applications in various branches of physics and mathematics; in particular, for construction of simple and approximation of solutions of various equations with variational methods.

The non-integer iterates provide functions with non-trivial asymptotic behaviour; for example, holomorphic function, that grows faster than any polynomial, but slower than any exponential. Analysis and implementation of superfunctions, abelfunctions and non-integer iterates of various functions greatly extends the arsenal of tools, available for the scientific research. For this reason, I think, the analysis of superfunctions, abelfunctions, and, in particular, functions SuNem and AuNem above, is important and interesting.

9 Conclusion

1. The 4th order polynomial $T(z)=\mathrm{Nem}_q(z)=z+z^3+qz^4$ by (1) for $q>0$ (figures 1,2) is considered as transfer function, from the point of view of its non-integer iterates. Function $T(z)=\mathrm{Nem}_q$ seemed to be difficult to treat with previously reported algorithms; at its fixed point $L=0$ (id est, $T(L)=L$), relations $T'(L)=1$, $T''(L)=0$, $T'''(L)\neq 0$ disable the previously published algorithms of evaluation of superfunction; so, the consideration is important.

2. The inverse function $\mathrm{ArqNem}_q=\mathrm{Nem}_q^{-1}$ is described; figures 1,2. At least in some vicinity of the positive part of the real axis, relation $\mathrm{Nem}_q\big(\mathrm{ArqNem}_q(z)\big)=z$ holds. Evaluation of the branch point as function of parameter q is described and implemented; eq. (22), figure 6. ArqNem_q is the only inverse function of the Nemtsov cundion

3. Treating $T=\mathrm{Nem}_q$ as transfer function, the superfunction SuNem_q by equation (29) and the corresponding abelfunction $\mathrm{uNem}_q=\mathrm{SuNem}_q^{-1}$ by (39) are constructed, eq.(29) and (40), figures 9, 10 and 11, 12. This is original part of this research. The complex double implementations of functions SuNem_q and AuNem_q are suggested. The algorithm returns of order of 14 decal digits and allow to plot the complex maps in real time.

4. Function $f=\mathrm{SuNem}_q$ by (29),(30)is the only entire solution, satisfying the additional condition $f(0)=1$ and bounded in the left hand side of the complex plane. Other solutions f of the same equation can be expressed with formula (45).

5. Superfunction $f=\mathrm{SuNem}_q$ and abelfucntin $g=\mathrm{uNem}_q$ allow to express the nth iterate of the transfer function $T=\mathrm{Nem}_q$ in the standard (for formalism of superfinctions) form: $T^n(z)=f(n+g(z))$; here, number n of iterates had no need to be integer. The analysis leads to

Conjecture 2: At least in some vicinity of positive values of m,n,z,q, the group relation holds, $T^{m+n}(z)=T^m(T^n(z))$.

5. Rigorous proof of the conjectures above can be matter for the further research. Method, used for the Nemtsov function, can be used to construct supercunctions and abelfunctions for other holomorphic functions with similar expansion at the fixed point (identity function, cubic term, highest terms). Superfunctions, abeldunctions and non-integer iterates greatly extend arsenal of tools for the scientific research and, in particular, approximation of functions. Consideration of the Nemtsov function is important part of the general research of constructing iterates of any holomorphic function.

6. The results above are obtained in attempts to suggest a holomorphic transfer function T such that its superfunction cannot be constructed. It will be interesting to invent at least one example of holomorphic function such that its iterates cannot be constructed in analogy with methods described.

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