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Method of random wave-vectors in the simulation of correlated random processes

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Summary. We describe a novel method for simulation of correlated homogeneous random processes with given second-order statistics of a general kind. These processes are specified by their correlation and cross-correlation functions. The simulated processes are constructed as sums of plane waves with random wave-vectors. For non-periodical processes with wide spectra, such a method requires much less memory, than the independent generation of the Fourier-components.

Die Methode der zufälligen Wellenvektoren zur Simulation von korrelierten Zufallsprozessen

Zusammenfassung. Es wird eine neue Methode der Simulation von korrelierten Zufallsprozessen mit vorgegebenen allgemeinen statistischen Eigenschaften zweiter Ordnung beschrieben. Solche Prozesse werden durch ihre Korrelations- und Kreuzkorrelationsfunktionen beschrieben. Der simulierte Prozeß wird durch eine Summe von ebenen Wellen mit zufälligen Wellenvektoren konstruiert. Für aperiodische Prozesse mit einem breiten Spektrum erfordert diese Methode wesentlich weniger Speicher als die unabhängige Generierung der Fourierkomponenten.

1. Introduction

Let us consider two correlated, homogeneous random processes $f(\vec{x})$ and $g(\vec{x})$, where $\vec{x}$ is N-dimensional vector. For example, to simulate the temporal behavior of two scalar parameters we set $N = 1$; to simulate the phase-amplitude distortions of a plane wave (KOUZNETSOV et al. 1997), we set $N = 2$; to simulate the 3-dimensional distribution of the real and imaginary parts of the refraction index in a random medium, we set $N = 3$, etc. Assume that the correlation functions

$$B_f(\vec{x}) = \langle f(\vec{x}_1)\, f(\vec{x}_2)\rangle\,, \quad B_g(\vec{x}) = \langle g(\vec{x}_1)\, g(\vec{x}_2)\rangle\,,$$
$$B_{fg}(\vec{x}) = \langle f(\vec{x}_1)\, g(\vec{x}_2)\rangle \qquad (1)$$

are given. While the processes are homogeneous, these are functions of the $\vec{x} = \vec{x}_2 - \vec{x}_1$ only; we may write, for ex., $B_f(\vec{x}) = \langle f(0)\, f(\vec{x})\rangle$.

The aim of this work is to develop an effective method to generate realizations of the correlated processes, reproducing the second-order statistics defined by Eq. (1). Such realizations can be used for the simulation of systems of atmospheric remote probes (KALLISTRATOVA and KON 1991), adaptive optics (ROGGERMANN et al. 1995), etc.

Here we consider only two correlated processes. The generalization for the case of many processes is straightforward.

A simple way to simulate the random process is to generate its Fourier-components, as it was done by COLES et al. (1995) and KOUZNETSOV and ORTEGA-MARTÍNEZ (1995). However, if the processes are not periodical, one has to use a grid of much larger size, than the size of the output sampling.

An alternative approach was applied by CANNON (1995) for regions of special (circular) form. This approach works quite well if one needs to simulate a single isotropic process; however, it is hard to apply for simulations of few correlated processes.

In order to avoid the above difficulties, we suggest a method in which, similar to the Fourier method, each realization is defined by its harmonics. But we will show that very few of them are enough to represent random or chaotic processes with spectra occupying many orders of magnitude. Such a reduction of the number of harmonics is possible due to their special choice. The wave-vectors of these harmonics are random. This is a reason to call such a way to generate random samples "method of Random WaveVectors" (RWV).

The validity of RWV for the simulation of correlated processes with given spectra was shown by KOUZNETSOV et al. (1997) in an application to atmospheric optics. It was shown that the structure functions are reproduced asymptotically exact: the relative deviations of the mean structure functions from the given functions decreases as $1/\sqrt{J}$, where J is number of samples considered. But it was not demonstrated, how few harmonics should be taken into account in the RWV to generate random or chaotic samples. In section 4 we present such a demonstration.

It should be mentioned that Cellular Automata (CA) could perhaps also provide very fast generation of chaotic samples (WALDROP 1992, TOFFOLI and MARGOLUS 1989). But it is not clear, how to make the CA generate random samples with given second-order correlations of general kind.

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We expect, the method of RWV is universal and can be efficiently used in many applications, from simulations of the temporal evolution of a scalar parameter (single-dimensional process) till the evolution of 3-dimensional fields (4-dimensional process). So, we keep the parameter N which specifies the dimension of simulated samples.

2. Generalized harmonics

We construct the processes as the sums of M plane waves (harmonics):

$$f(\vec{x}) = \sum_{m=1}^{M} F(\vec{p}_m) \cos(\vec{p}_m\vec{x} + \varphi_m), \tag{2}$$

$$g(\vec{x}) = \sum_{m=1}^{M} G(\vec{p}_m) \cos(\vec{p}_m\vec{x} + \varphi_m + \psi_m)$$

These sums are similar to the Fourier-series, but we don't assume that $\vec{p}_m$ belongs to some definite grid, as wave-vectors used in the Fast-Fourier Transform (FFT).

We treat $\vec{p}_m$ as random vectors, and φ_m, ψ_m as random numbers. All harmonics should be generated independently. Phases φ_m are distributed uniformly and totally independent. This implies the homogeneity of the processes. As for the phases ψ_m, they may be entangled with wave-vectors $\vec{p}_m$ of the same harmonics.

Functions F and G also may be random, but this makes the algorithm more complicated. During the simulations with $N = 2$ (KOUZNETSOV et al. 1997), there was seen no visual difference between samples generated with regular F, G and samples generated with a Gaussian distribution of F, G. (As for the second-order statistics, it is reproduced exactly in both cases.) So, the randomness of wave-vectors is quite enough to produce chaotic samples, without making the coefficients F and G random. Thus, we treat these coefficients as regular non-negative functions of an N-vector argument.

We should express F, G in terms of given functions B defined by Eq. (1). Substitution of (2) into (1) gives:

$$B_f(\vec{x}) = \sum_{m=1}^{M}\sum_{n=1}^{M} \left\langle F(\vec{p}_m) F(\vec{p}_n) \cos(\vec{p}_m\vec{x}_1 + \varphi_m) \cos(\vec{p}_n\vec{x}_2 + \varphi_n) \right\rangle, \tag{3}$$

$$B_g(\vec{x}) = \sum_{m=1}^{M}\sum_{n=1}^{M} \left\langle G(\vec{p}_m) G(\vec{p}_n) \cos(\vec{p}_m\vec{x}_1 + \varphi_m + \psi_m) \cos(\vec{p}_n\vec{x}_2 + \varphi_n + \psi_n) \right\rangle, \tag{4}$$

$$B_{fg}(\vec{x}) = \sum_{m=1}^{M}\sum_{n=1}^{M} \left\langle F(\vec{p}_m) G(\vec{p}_n) \cos(\vec{p}_m\vec{x}_1 + \varphi_m) \cos(\vec{p}_n\vec{x}_2 + \varphi_n + \psi_n) \right\rangle. \tag{5}$$

Because φ_m are distributed independently and uniformly between $(-\pi, \pi]$, we have the only single sums:

$$B_f(\vec{x}) = \sum_{m=1}^{M} \left\langle F(\vec{p}_m) F(\vec{p}_m) \cos(\vec{p}_m\vec{x}_1 + \varphi_m) \cos(\vec{p}_m\vec{x}_2 + \varphi_m) \right\rangle, \tag{6}$$

$$B_g(\vec{x}) = \sum_{m=1}^{M} \left\langle G(\vec{p}_m) G(\vec{p}_m) \cos(\vec{p}_m\vec{x}_1 + \varphi_m + \psi_m) \cos(\vec{p}_m\vec{x}_2 + \varphi_m + \psi_m) \right\rangle, \tag{7}$$

$$B_{fg}(\vec{x}) = \sum_{m=1}^{M} \left\langle F(\vec{p}_m) G(\vec{p}_m) \cos(\vec{p}_m\vec{x}_1 + \varphi_m) \cos(\vec{p}_m\vec{x}_2 + \varphi_m + \psi_m) \right\rangle. \tag{8}$$

We apply $\cos(u)\cos(v) = (\cos(u+v) + \cos(u-v))/2$ and use the uniform distribution of φ_m; it gives:

$$B_f(\vec{x}) = \frac{1}{2}\sum_{m=1}^{M} \left\langle F(\vec{p}_m)^2 \cos(\vec{p}_m\vec{x}) \right\rangle,$$

$$B_g(\vec{x}) = \frac{1}{2}\sum_{m=1}^{M} \left\langle G(\vec{p}_m)^2 \cos(\vec{p}_m\vec{x}) \right\rangle, \tag{9}$$

$$B_{fg}(\vec{x}) = \frac{1}{2}\sum_{m=1}^{M} \left\langle F(\vec{p}_m) G(\vec{p}_m) \cos(\vec{p}_m\vec{x} + \psi_m) \right\rangle =$$

$$= \frac{1}{2}\sum_{m=1}^{M} \left\langle F(\vec{p}_m) G(\vec{p}_m) \cos(\vec{p}_m\vec{x}) \cos(\psi_m) \right\rangle. \tag{10}$$

In the last equation, we applied $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$ and supposed that $\langle\sin\psi_m\rangle = 0$ for all $\vec{p}_m$.

Let all wave-vectors $\vec{p}_m$ be distributed with density $\mu(\vec{p}_m)$; so we have

$$B_f(\vec{x}) = \frac{1}{2}\sum_{m=1}^{M} \int \mu(\vec{p}_m) F(\vec{p}_m)^2 \cos(\vec{p}_m\vec{x})\, d^N\vec{p}_m = \frac{M}{2}\int \mu(\vec{p}) F(\vec{p})^2 \cos(\vec{p}\vec{x})\, d^N\vec{p}, \tag{11}$$

$$B_g(\vec{x}) = \frac{1}{2}\sum_{m=1}^{M} \int \mu(\vec{p}_m) G(\vec{p}_m)^2 \cos(\vec{p}_m\vec{x})\, d^N\vec{p}_m = \frac{M}{2}\int \mu(\vec{p}) F(\vec{p})^2 \cos(\vec{p}\vec{x})\, d^N\vec{p}, \tag{12}$$

$$B_{fg}(\vec{x}) = \frac{1}{2}\sum_{m=1}^{M} \int \mu(\vec{p}_m)\, \eta(\vec{k}_m, \psi_m) F(\vec{p}_m) G(\vec{p}) \cos(\vec{p}_m\vec{x}) \cos(\psi_m)\, d^N\vec{p}_m d\psi_m = \frac{M}{2}\int \mu(\vec{p})\, \eta(\vec{p}, \psi) F(\vec{p}) G(\vec{p}) \cos(\vec{p}\vec{x})\, d^N\vec{p} d\psi, \tag{13}$$

where $\eta(\vec{k}_m, \psi_m)$ is the conditional density of the probability of the distribution of phase ψ_m. Thus, $\mu(\vec{p}_m)\,\eta(\vec{p}_m, \psi_m)$ is the joint probability density of the wave-vector $\vec{p}_m$ of the m-th harmonic and relative phase ψ_m.

Taking the Fourier-transforms

$$b(\vec{k}) = (2\pi)^{-N/2} \int e^{i\vec{k}\vec{x}} B(\vec{x})\, d^N\vec{x} \tag{14}$$

of (11)–(13), we have:

$$b_f(\vec{k}) = (2\pi)^{N/2} \frac{M}{2}\left(\frac{1}{2}\mu(-\vec{k}) F(-\vec{k})^2 + \frac{1}{2}\mu(\vec{k}) F(\vec{k})^2\right), \tag{15}$$

$$b_g(\vec{k}) = (2\pi)^{N/2}\frac{M}{2}\left(\frac{1}{2}\mu(-\vec{k})\,G(-\vec{k})^2 + \frac{1}{2}\mu(\vec{k})\,G(\vec{k})^2\right), \quad (16)$$

$$b_{fg}(\vec{k}) = (2\pi)^{N/2}\frac{M}{2}\left(\frac{1}{2}\int\mu(-\vec{k})\,\eta(-\vec{k},\psi)\,F(-\vec{k})\,G(-\vec{k}) \times\cos(\psi)\,d\psi + \frac{1}{2}\int\mu(\vec{k})\,\eta(\vec{k},\psi)\,F(\vec{k})\,G(\vec{k})\cos(\psi)\,d\psi\right). \quad (17)$$

To get (15)–(17), we apply $\cos(a) = (e^{ia} + e^{-ia})/2$; then use the spectral representation of δ-function: $\int \exp(i\vec{b}\vec{x})d^N\vec{x} = (2\pi)^N\delta^N(\vec{b})$. Due to δ-functions, no integrals appear in (15)–(17); due to the coefficient $(2\pi)^N$ before the δ-function, we have coefficients $(2\pi)^{N/2}$ instead of coefficient $(2\pi)^{-N/2}$ in Eq. (14).

Assuming that $\mu(-\vec{k}) = \mu(\vec{k})$, $\eta(-\vec{k},\psi) = \eta(\vec{k},\psi)$, $F(-\vec{k}) = F(\vec{k})$, and $G(-\vec{k}) = G(\vec{k})$, we have:

$$b_f(\vec{k}) = (2\pi)^{N/2}\frac{M}{2}\mu(\vec{k})\,F(\vec{k})^2,$$
$$b_g(\vec{k}) = (2\pi)^{N/2}\frac{M}{2}\mu(\vec{k})\,G(\vec{k})^2, \quad (18)$$

$$b_{fg}(\vec{k}) = (2\pi)^{N/2}\frac{M}{2}\mu(\vec{k})\,F(\vec{k})\,G(\vec{k})\int\cos(\psi)\,\eta(\vec{k},\psi)\,d\psi. \quad (19)$$

So, to reproduce the given second-order statistics, we choose any μ, η, F, G such that

$$F(\vec{k})^2\mu(\vec{k}) = \frac{2}{M(2\pi)^{N/2}}b_f(\vec{k}),$$
$$G(\vec{k})^2\mu(\vec{k}) = \frac{2}{M(2\pi)^{N/2}}b_g(\vec{k}), \quad (20)$$

$$F(\vec{k})\,G(\vec{k})\,\mu(\vec{k})\int\eta(\vec{k},\psi)\,d\psi = \frac{2}{M(2\pi)^{N/2}}b_{fg}(\vec{k}). \quad (21)$$

The right parts of equations (20) should be non-negative. Also, it is necessary that $|b_f(\vec{k})b_g(\vec{k})| \geq b_{fg}(\vec{k})^2$: the cross-correlation cannot be greater than the mean geometric value of the correlations.

The distribution μ and functions F and G may be chosen before taking into account the cross-correlation. Note that we have 2 equations (20) for 3 functions (F, G, and μ); it allows some freedom. We may choose any positive μ, then Eq. (20) determines F and G. It is natural to make $\mu(\vec{k})$ be greater at wavenumbers $|\vec{k}|$ which are more important for the problem to be simulated. For example, we may take $\mu(\vec{k}) = a\sqrt{b_f(k)^2 + b_g(k)^2}$, ($a$ = const.), to make the similated samples looking more chaotic at relatively small number M of harmonics in the composition (2) of simulated processes.

If we cannot predict what wavenumbers are of importance for the problem, and if wavenumbers occupy many orders of magnitude, the logarithmic scale is suitable. Consider the new variable $y = \log(|\vec{p}_m|/K_0)$. Let it be distributed uniformly from $\log(K_1/K_0)$ till $\log(K_2/K_0)$; and let orientations of the wave-vectors be distributed uniformly (K_0, K_1, K_2 are constants). This leads to the density

$$\mu(\vec{k}) = (S_N|\vec{k}|^N\log(K_2/K_1))^{-1}; \quad (22)$$

where S_N is the area of a N-dimensional sphere of radius 1: $S_1 = 2$, $S_2 = 2\pi$, $S_3 = 4\pi$,..; $S_{n+1} = S_n\int_0^\pi \sin(\gamma)^{n-1}d\gamma$.

Of course, any other density μ can also be used. It causes no problem, to generate pseudo-random numbers with given density. There are effective algorithms for pseudo-random numbers distributed uniformly between [0,1) (WALDROP 1992).

To construct the random variable r distributed with density $p(r)$, we consider some new variable $t = T(r)$, where T is monotonous function. Then, for any function $h = H(r)$, we may write

$$\langle h\rangle = \langle H(r)\rangle = \int H(r)\,\rho(r)\,dr = \int H(R(t))\,\rho(R(t))\,R'(t)\,dt = \int H(R(t))\,\tau(t)\,dt, \quad (23)$$

where $R = T^{-1}$ is the inverse function; R′ denotes its derivative. While H is arbitrary, the distribution density of variable t is $\tau(t) = \rho(R(t))\,R'(t)$. If we want the new variable t be distributed uniformly from 0 to 1, we take $\tau(t) = 1$. Then the function R can be obtained from the equation $R'(t) = 1/\rho(R(t))$. For simple densities ρ, such differential equation can be resolved in quadratures. (It is important for efficient simulations to have a fast algorithm to calculate all random variables).While t is distributed uniformly, $R(t)$ is distributed with given density ρ.

To reproduce the cross-correlation, we should choose any distribution η of ψ_m at $\vec{p}_m = \vec{k}$, such that

$$\int\eta(\vec{k},\psi)\cos(\psi)\,d\psi = \frac{b_{fg}(\vec{k})}{\sqrt{b_f(\vec{k})\,b_g(\vec{k})}} \quad (24)$$

To satisfy Eq. (24), we may choose the Gaussian distribution,

$$\eta(\vec{k},\psi) = \frac{1}{2\sqrt{\pi\alpha(\vec{k})}}\exp\left(-\frac{\left(\psi + \theta(-b_{fg}(\vec{k}))\,\pi\right)^2}{4\alpha(\vec{k})}\right),$$
$$\alpha(\vec{k}) = \log\frac{\sqrt{b_f(\vec{k})\,b_g(\vec{k})}}{|b_{fg}(\vec{k})|}, \quad (25)$$

where $\theta(z) = \begin{cases}0, & z<0\\ 1, & z\geq 0\end{cases}$.

If $b_{fg}(\vec{k}) = 0$, for ψ_m we take the uniform distribution between $(-\pi, \pi]$. Substituting Eq. (25) into Eq. (24) and taking the Gaussian integral, one can see that such choice reproduces the correct cross-correlation.

Now we have defined all variables which appear in Eq. (2).

3. Algorithm

The formulas of the previous section permit us to construct the algorithm to simulate random correlated processes.

In some application, only the structure functions $D_f(\vec{x}) = \langle(f(\vec{x}_1)-f(\vec{x}_2))^2\rangle$, $D_g(\vec{x}) = \langle(g(\vec{x}_1)-g(\vec{x}_2))^2\rangle$, $D_{fg}(\vec{x}) = \langle(f(\vec{x}_1)-f(\vec{x}_2))(g(\vec{x}_1)-g(\vec{x}_2))\rangle$ are given. While the processes should be simulated in a finite region, the smooth truncation of the structure function at the relative separation equal to the maximal size of this region is appropriate. This permits us to construct the correlation functions B, with spectra b satisfying the conditions

$$b_f(\vec{k}) \ge 0, \quad b_g(\vec{k}) \ge 0, \quad \left|b_{fg}(\vec{k})\right| \le \sqrt{b_f(\vec{k})\, b_g(\vec{k})}. \tag{26}$$

If any of (26) does not apply, such a process cannot be realized. For example, the structure function $D_f(\vec{x}) = |\vec{x}|^{7/3}$ is forbidden; no smooth truncation gives B_f with non-negative Fourier-transform b_f, and there exists no homogeneous process with such structure function.

If conditions (26) are satisfied, we choose the density μ of the wave-vectors. If there is no special indication what scale is essential in the simulated processes, the uniform distribution of the logarithm of the wavenumber and random orientation of wave-vectors leading to formula (22), seems to be most safe. Constants K_1 and K_2 should be chosen from physical reasons. For example, for waves in the atmosphere, there is no reason to take $1/K_2$ less than the size of the molecule, and there is no reason to take $1/K_1$ greater than the size of the planet. Thus, effective spectra of all homogeneous processes in geophysics are always limited.

The conditional density $\eta(\vec{k}, \psi)$ must satisfy Eq. (24). A simple choice is defined by formula (25).

The number M of harmonics in Eq. (2) should be chosen too. Eq. (2) gives the correct second-order statistics even at $M = 1$, but we expect the realizations to be in some sense chaotic. The example below shows that we do not need to take a lot of harmonics to produce chaotic realizations. Usually, it is enough to take $5N$ harmonics for each decade of effective wavenumbers; then no visual regularities appear in the simulated random samples.

At given M, N and μ, functions F and G are defined by Eq. (20).

Now everything is ready for simulation. For $m = 1..M$, we generate random wave-vectors $\vec{k}_m$ with density μ and random φ_m; then, at given $\vec{k}_m$, we generate ψ_m with density $\eta(\vec{k}, \psi)$. These three arrays (φ, $\vec{k}$ and ψ) define the random samples. We have no special algorithm (like FFT) for sums in Eq. (2); so, to construct this sample at any grid (it may be rectangular as well as curvilinear), we apply directly the formulas (2) and (20).

4. Example

As a demonstration, let us simulate a 2-dimensional isotropic process $f(\vec{x})$ in a region $|\vec{x}| < 1$ with the structure function

$$D_f(\vec{x}) = D(\vec{x}) = |\vec{x}|^{4/3}, \quad |\vec{x}| < 1. \tag{27}$$

The appropriate correlation function can be constructed as:

$$B(\vec{x}) = B(|\vec{x}|) = \begin{cases} .937 - 0.5|\vec{x}|^{4/3}, & |\vec{x}| \le 1 \\ .937 \exp(-0.762\vec{x}^2), & |\vec{x}| \ge 1 \end{cases} \tag{28}$$

The coefficients provide the coupling of the function $B(r)$ as well as of its derivative at the point $r = 1$.)

The Fourier-transform b of the function (28) can be calculated as the Bessel-transform of the function B:

$$b(\vec{k}) = \frac{1}{2\pi} \int d^N \vec{x}\, e^{i\vec{k}\vec{x}}\, B(\vec{x}) = \int_0^\infty r dr\, J_0(|\vec{k}|r) B(r), \tag{29}$$

where J_0 is the Bessel function.

The transform (29) is practically positive, as we require. (We substitute small negative values of the right part of (29) to zero; then, the deviation of the inverse transform from the function defined by (28) is very small, it couldn't be seen at graphics.)

But the function $b(\vec{k})$ decays slowly at large $|\vec{k}|$, due to the peculiarity at $\vec{k}$=0. We need about 4 orders of magnitude of variation of $|\vec{k}|$ to represent the structure function with at least 3 decimal digits.

Taking the uniform distribution of the logarithm of wavenumbers between log(.01) and log(100), we apply the RWV. In each of Fig. 1, 2, 3 we show the sequence of 8 independent realizations of the random samples, each one corresponds to the square of size 1.28×1.28; the grid 64×64 with step $\Delta x = 0.02$ was used to plot them. Samples in Fig. 1 were calculated with $M = 10$; in Fig. 2 with $M = 50$ and with $M = 200$ in Fig. 3. Samples are displayed in the sequence of the generation, no selection was applied.

Samples in each of Fig. 1, 2, 3 reproduce the same structure function (27), but samples in Fig. 1 cannot be considered as random, we see quasi-regular structures. These structures don't appear in Fig. 2, 3. No difference in the structure is seen between Fig. 2 and Fig. 3. This shows, that even $M = 50$ harmonics in the sums in (2) are enough to simulate random samples. The computation of all figures took few minutes at the PC with the 486 processor. As it was mentioned, the slowest operation is the summation in Eq. (2). With $M = 50$, it includes the calculation and summation of about $64 \times 64 \times 50 \approx 200\,000$ terms/sample.

To represent the spectra of random samples in this example on a uniform grid, we would need about $10\,000 \times 10\,000$ points, that would be too much for our computer. With some renormalization of the Fourier-components, for grids of a given step, the size of the grid may be reduced. At the rough grid in the configurational space, the highest harmonics act like a δ-correlated noise. To reproduce such a noise, we have no need to take more Fourier-harmonics than the amount of points in the final grid. But even in this case, for samples of 64×64, we would need to take the grid at least till 256×256, to hide the periodicity. Then the calculation of the FFT would take about $256^2 \log_2(256^2) \approx 1\,000\,000$ operations, like in the RWV case above.

From the point of view of the computational time, both methods are similar.

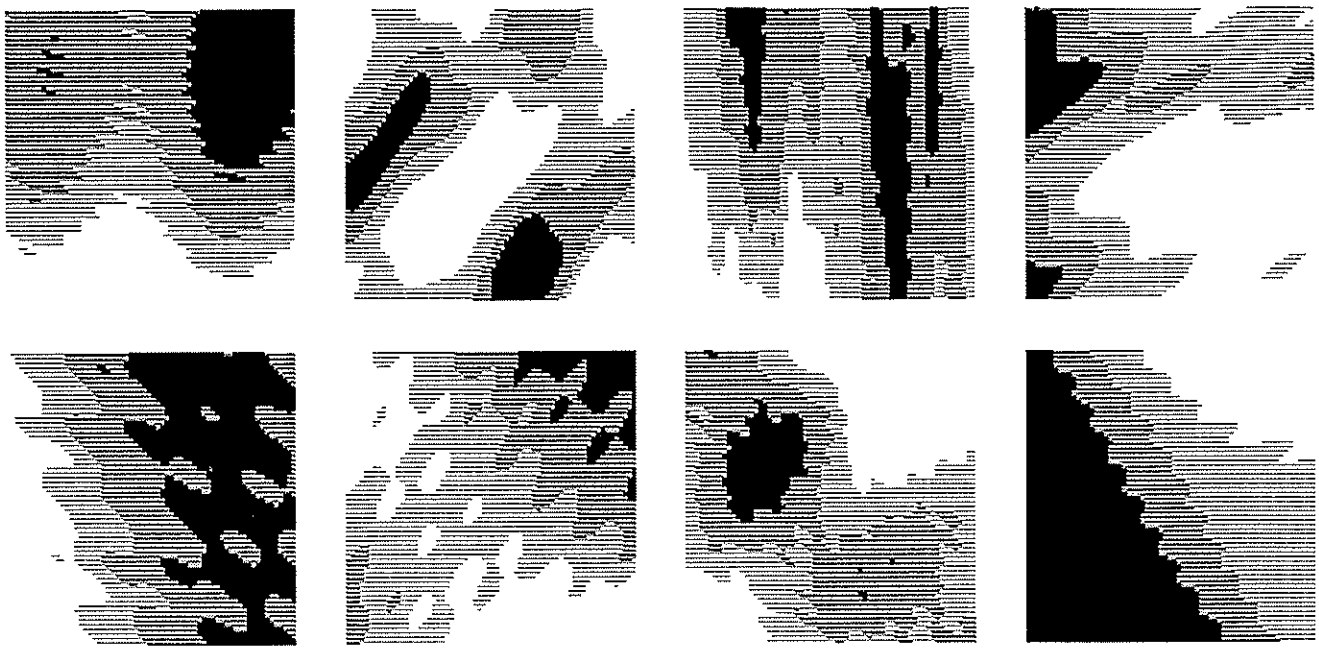

Fig. 1. 8 random samples with the correlation function (28) constructed on a 64 × 64 grid with steps $\Delta x = 0.02$ by the RWV for $M = 10$; some quasi-regular structures are seen.

Abb. 1. 8 Zufallsmuster mit der Korrelationsfunktion (28) auf einem Gitter 64 × 64 mit einem Maschenabstand von $\Delta x = 0{,}02$ konstruiert durch RWV bei $M = 10$, einige quasi-regelmäßige Texturen sind sichtbar.

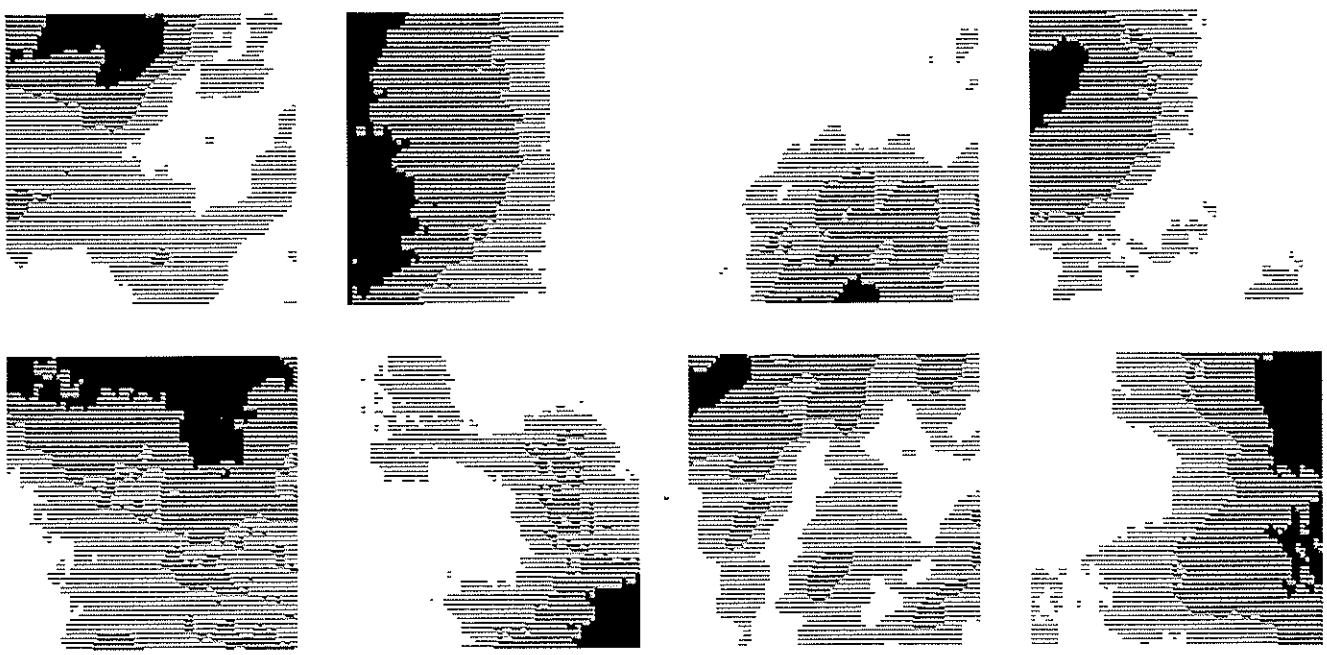

Fig. 2. 8 random samples with the correlation function (28) constructed on a 64 × 64 grid with step $\Delta x = 0.02$ by the RWV for $M = 50$; no quasi-regular structures are seen.

Abb. 2. 8 Zufallsmuster mit der Korrelationsfunktion (28) auf einem Gitter 64 × 64 mit einem Maschenabstand von $\Delta x = 0{,}02$ konstruiert durch RWV bei $M = 50$, es sind keine quasi-regelmäßigen Texturen sichtbar.

Another situation occurs with the memory requirements. To define the sample, in the method of RWV it is enough to store $M \times (N + 1) = 150$ numbers, while for the Fourier-method, $2 \times (4L)^2 \approx 128\,000$ numbers are necessary, where $L = 64$ is the amount of steps of the grid.

The difference is even more pronounced, if we would generate larger samples. For samples of 300 × 300, with $M = 100$, we need about 10^7 operations with RWV, or with the FFT-composition with a grid of 1024 × 1024. (As we mentioned, this excess hides the periodicity.) But, for two

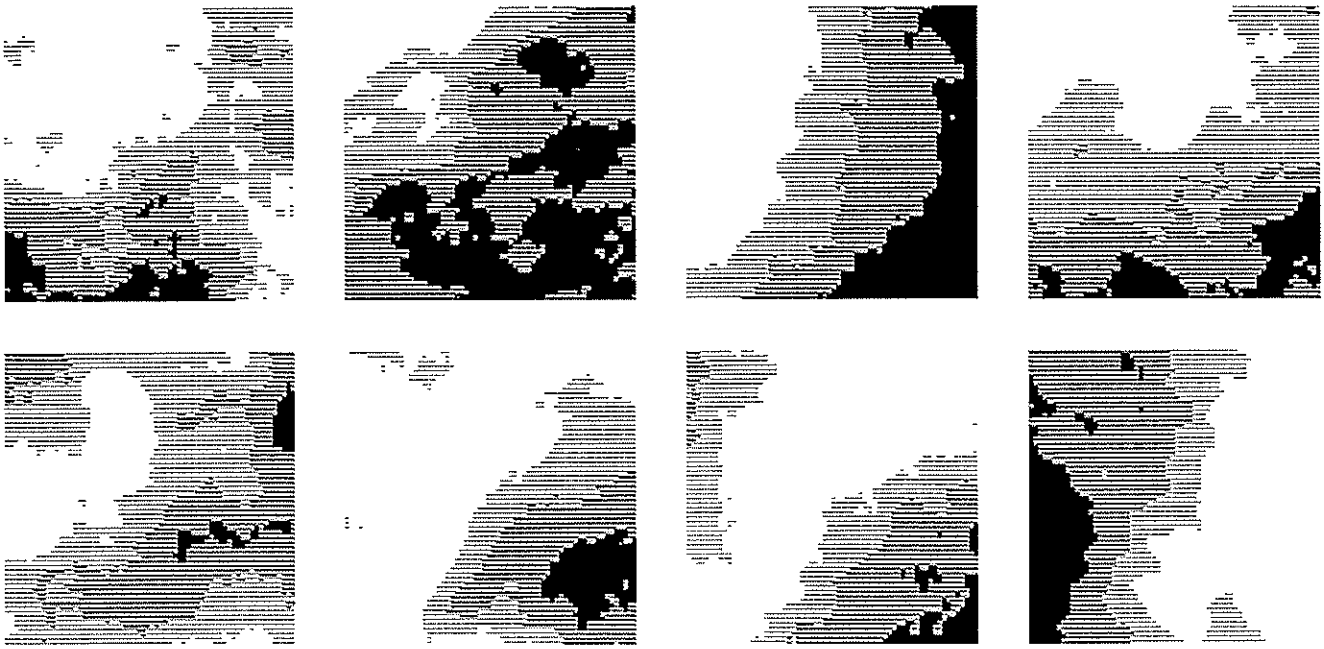

Fig. 3. 8 random samples with the correlation function (28) constructed on a 64 × 64 grid with steps $\Delta x = 0.02$ by the RWV for $M = 200$; no difference in the structure from Fig. 2 is seen.

Abb. 3. 8 Zufallsmuster mit der Korrelationsfunktion (28) auf einem Gitter 64 × 64 mit einem Maschenabstand von $\Delta x = 0{,}02$ konstruiert durch RWV bei $M = 200$, in der Textur ist kein Unterschied zu Abb. 2 zu sehen.

correlated processes, we need to keep about 400 numbers with the RWV (100 two-vectors p_m, m = 1..100 and 200 scalars φ_m, $m = 1..100$ and ψ_m, $m = 1..100$); and we need about 2×10^6 numbers with the FFT-composition. In this case, the RWV needs 5000 times less memory than the FFT-composition.

We expect, for multidimensional processes ($N > 2$) with fine grids, the RWV will still work while the FFT-composition gets uncomputable.

We have no proof that 50 harmonics in (2) "are enough". This could depend on the specific application. So, we present only visual qualitative observations to support our choice.

On the other hand, the finite number M of harmonics could be used to adjust some higher-correlation properties of simulated processes. Sometimes, the clouds in the sky form quasiperiodical structures like some of the samples in Fig. 1. So, may be, the small number $M < 50$ could be better for reproducing some higher-order correlations. Authors would greatly appreciate any experimental data about any third- and fourth-order correlations in the atmosphere.

Note, that if we have no need to reproduce the given second-order correlations, CA could generate chaotic samples even more effective (see, for ex., WALDROP 1992). We see no structure difference between samples in Fig. 2, 3 and the samples generated by the Genetic Drift, presented in Fig. 9.4(a) by TOFFOLI and MARGOLUS (1989).

5. Conclusion

A novel method, RWV, for the simulation of correlated random processes with given second-order statistics is proposed. The simulated processes are constructed as sums of harmonics with random wave-vectors. The set of wave-vectors is the same for both processes. The phases for different harmonics are independent, but are correlated for the same harmonics of different processes. This correlation is defined by the Fourier-transform of the cross-correlation function.

With the RWV method, the random processes can be constructed on any grid, not only on a rectangular one. The spectra of the simulated processes may occupy many orders of magnitude without memorizing large grids.

The comparison of the efficiency of RWV with the FFT composition shows the similar computational time, but the memory requirements with RWV are some decades less. This should be considered as advantage of our method.

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