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The decay of a three-level atom and time–frequency oscillations

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Abstract. The equations of motion of the simple quantum mechanical system are solved. The large-time limit of the solution gives the mutual wave function of two output photons. In the momentum presentation for the first photon and the coordinate presentation for the second, this function oscillates. Such oscillations may be observed using the simultaneous counting scheme.

1. Introduction and discussion

The aim of this paper is to show an unexpected property of the two-photon state produced by the decay of a three-level atom. Let us consider the system with the Hamiltonian $H = H_0 + H_1$,

$$H_0 = \int a_k^+ a_k k \, dk + E u^+ u + F v^+ v, \tag{1}$$

$$H_1 = \int \xi (a_k^+ v^+ u + a_k u^+ v) \, dk + \int \zeta (a_k^+ w^+ v + a_k v^+ w) \, dk, \tag{2}$$

where

$$\begin{aligned} [a_k, a_p^+] &= \delta(k-p), \qquad uu^+ + u^+ u = 1, \\ vv^+ + v^+ v &= 1, \qquad ww^+ + w^+ w = 1, \end{aligned} \tag{3}$$

and all other commutators of a, a^+, u, u^+, v, v^+, w, w^+ are equal to zero. The Hamiltonian H describes a three-level atom interacting with the continuum of modes of the quantum field; each of these modes may be considered as a harmonic oscillator. The scheme of levels of the atom is presented in figure 1. In (2) ξ and ζ are coupling constants. Later it will become clear that $\Gamma_1 = 2\pi\xi^2$ and $\Gamma_2 = 2\pi\zeta^2$ are rates of decay of levels u and v; Γ_1 and Γ_2 are supposed to be small in comparison with energies E, F. In this assumption the Schrödinger equation

$$\mathrm{i}\, d\psi/dt = \mathrm{i}\dot{\psi} = H\psi, \tag{4}$$

may be solved by the Weiscopf method described in [1]. Let us treat the solution of (4) in the form

$$\psi = h \exp(-\mathrm{i}Et)\, u^+|0\rangle + \int g_k \exp[-\mathrm{i}(F+k)t]\, v^+ a_k^+ |0\rangle \, dk + \int\int f_{k,p} \exp[-\mathrm{i}(k+p)t]\, a_p^+ a_k^+ w^+ |0\rangle \, dp \, dk, \tag{5}$$

where $|0\rangle$ is the vacuum, $a_k|0\rangle=u|0\rangle=v|0\rangle=w|0\rangle=0$.

Substitution of (5) into (4) gives the system

$$\mathrm{i}\dot{h}\exp(-\mathrm{i}Et)=\int \xi g_k \exp[-\mathrm{i}(F+k)t]\,\mathrm{d}p, \tag{6a}$$

$$\mathrm{i}\dot{g}_k \exp[-\mathrm{i}(F+k)t]=\int \zeta(f_{k,q}+f_{q,k})\exp[-\mathrm{i}(k+q)t]\,\mathrm{d}q+\xi h\exp(-\mathrm{i}Et), \tag{6b}$$

$$\mathrm{i}\dot{f}_{k,p}\exp[-\mathrm{i}(k+p)t]=\zeta g_k\exp[-\mathrm{i}(F+p)t]\mu+\zeta g_p\exp[-\mathrm{i}(F+k)t](1-\mu), \tag{6c}$$

where μ is an arbitrary parameter. We may adopt the equation $f_{k,p}=f_{p,k}$; and to coordinate it with (6 c) we must set $\mu=\frac{1}{2}$.

The decay from the upper level u corresponds to the initial conditions at $t=0$:

$$h=1, \qquad g_k=f_{k,p}=0. \tag{7}$$

Let us suppose that there is no degeneration, that is

$$|E-2F|\gg\xi^2, \qquad |E-2F|\gg\zeta^2.$$

In this approach the solution of (6), (7) is

$$h(t)=\exp(-\pi\xi^2 t), \tag{8}$$

$$g_k(t)=-\mathrm{i}\xi\frac{\exp[\mathrm{i}(k-E+F)t-\pi\xi^2 t]-\exp[-\pi\zeta^2 t]}{\mathrm{i}(k-E+F)+\pi\zeta^2-\pi\xi^2}, \tag{9}$$

$$f_{k,p}=\tfrac{1}{2}\tilde{f}_{k-E+F,p-F}(t)+\tfrac{1}{2}\tilde{f}_{p-E+F,k-F}(t), \tag{10}$$

where

$$\tilde{f}_{q,r}(t)=-\frac{\xi}{(\mathrm{i}q+\pi\zeta^2-\pi\xi^2)}\left(\frac{\exp[(\mathrm{i}q+\mathrm{i}r-\pi\xi^2)t]-1}{\mathrm{i}(q+r)-\pi\xi^2}-\frac{\exp[(\mathrm{i}r-\pi\zeta^2)t]-1}{\mathrm{i}r-\pi\zeta^2}\right). \tag{11}$$

The distribution of emitted photons is determined by the limit $t\to\infty$: $h=g_k=0$,

$$\tilde{f}_{q,r}=\lim_{t\to\infty}\tilde{f}_{q,r}(t)=\xi\zeta/[(\mathrm{i}q+\mathrm{i}r-\pi\xi^2)(\mathrm{i}r-\pi\zeta^2)]. \tag{12}$$

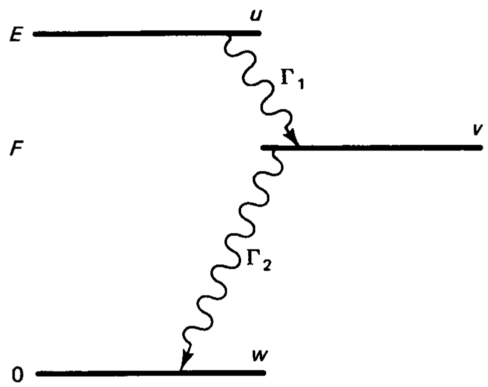

Figure 1. The energetic levels of the atom.

This formula gives the mutual distribution of two photons in the momentum presentation. $|\tilde{f}_{q,r}|^2$ can be measured by a spectral device. We can measure the square of modulus of the Fourier transform

$$f(x,y)=\frac{1}{2\pi}\int \exp(-iqx-iry)\tilde{f}_{q,r}\,dq\,dr$$
$$=2\pi\xi\zeta\exp[-\pi\zeta^2(y-x)-\pi\xi^2 x]\,\theta(x)\,\theta(y-x), \tag{13}$$

where $\theta(x)=\{0 \text{ if } x<0, 1 \text{ if } x>0\}$, by registration of moments when the two photons reach the detectors.

The sense of (13) is very simple: the upper level (u) decays first, at a rate $\Gamma_1=2\pi\xi^2$, and after it the second level (v) decays at a rate $\Gamma_2=2\pi\zeta^2$. We may test that the normalization is not lost:

$$\int\int |\tilde{f}_{q,r}|^2\,dq\,dr=\int\int |f(x,y)|^2\,dx\,dy=1.$$

More interesting is the case when the first photon with a frequency approximately $E-F$ goes to a spectral device while the second one (frequency approximately F) goes directly to the counter. By a set of experiments this system measures the square of the modulus of the function $f_q(y)$:

$$|f_q(y)|^2=\left|\frac{1}{\sqrt{2\pi}}\int \exp(-iry)\tilde{f}_{q,r}\,dr\right|^2$$
$$=\frac{2\pi\xi^2\zeta^2}{[q^2+(\pi\zeta^2-\pi\xi^2)^2]}\theta(y)\{\exp(-2\pi\xi^2 y)$$
$$+\exp(-2\pi\zeta^2 y)-2\exp[-\pi(\zeta^2+\xi^2)y]\cos(qy)\}. \tag{14}$$

In the case $\xi=\zeta$ $(\Gamma_1=\Gamma_2=\Gamma)$ this formula reduces to

$$|f_q(y)|^2=(\Gamma/q)^2\exp(-\Gamma y)[1-\cos(qy)]\theta(y). \tag{15}$$

This function is presented in figure 2. The period of the oscillations is determined by the frequency shift $q=k-(E-F)$ of the first photon.

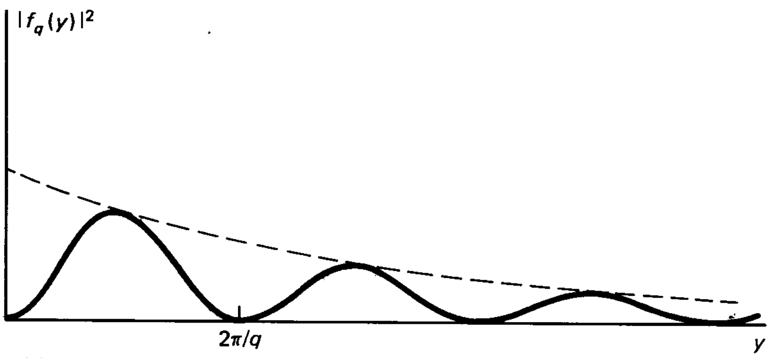

Figure 2. The probability density of registration of the second photon to be at a time y when the first photon has the frequency shift q. The continuous curve represents the case $\Gamma_1=\Gamma_2=\Gamma$ by formula (15); the dashed curve corresponds to the trend $2(\Gamma/q)^2\exp(-\Gamma y)$.

The oscillating behaviour of the two-photon wave function may be explained by the semiclassical speculation. Let us adopt that the reduction of the wave packet happens when the second photon is registered at the time y. Hence, the first photon was emitted during the finite time. The respective classical oscillator emits the wave packet during the time interval $0<t<y$ and has the amplitude of the Fourier component (with frequency shift q) proportional to

$$c_q(y)=\int_0^y \exp(\mathrm{i}qt)\,\mathrm{d}t=\frac{1-\exp(\mathrm{i}qy)}{\mathrm{i}q}. \tag{16}$$

The classical spectrum $|c_q(y)|^2$ given by (16) contains the same oscillating factor $[1-\cos(qy)]$ as that given by quantum theory (formula (15)).

A few words about the preparing of the atom of the u state. It may be done either by two short π-pulses of frequencies F and $E-F$ or by relaxation from the fourth (higher than u) level with the compulsory registration of the photon emitted by the relaxation (this photon must switch on the timer).

2. Conclusion

The presented calculations show that multi-photon measurements of emitted quanta may give non-trivial dependencies. At the two-photon decay the probability density of the moment of registration of the second photon depends on the colour (i.e. on its frequency shift relative to the middle of the spectral line) of the first one. It may be detected by the simultaneous scheme. It would be interesting to see oscillations like those in the figure 2 in an experiment with a trapped ion.

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References

[1] Weisscopf, V., and Wigner, E., 1930, *Z. Physik*, **63**, 54–73.

[2] H. J. Kimble, A. Mezzacappa. Time dependence of photon correlations in a 3-levels atomic cascade. – Phys. Rev. A 31, N6, P. 3686–3697 (1985)