*Владикавказский математический журнал*
*2010, Том 12, Выпуск 2, С. 31–45*

*УДК* **519.688, 517.18**

# TETRATIONAL AS SPECIAL FUNCTION

**D. Kouznetsov**

Holomorphic tetrational (superexponential) to base $e$ and its inverse function (arctetrational) are approximated with elementary functions.

**Mathematics Subject Classification (2000):** 65Q20, 30B40, 41A10, 33E99, 90-04, 39B12.

**Key words:** tetrational, superfunction, Abel equation, Abel function, holomorphic extension, analytic continuation, numerical implementation, special functions, Taylor expansion, superexponential, arctetrational, superlogarithm, non-integer iteration.

## 1. Introduction

The terational and the superexponential refer to a holomorphic solution $F$ of the equation

$$F(z+1) = \exp_b\big(F(z)\big). \tag{1}$$

Such equation is considered since years 1950 [1–10]; in particular, for the natural base $b = \mathrm{e}$. Name «superexponential» indicates, that function $F$ is *superfunction* [9,11] of the exponential.

In general, for some function $H$, which can be called also the *transfer function* [9, 11, 12], a superfunction $F$ is a holomorphic solution of the equation

$$F(z+1) = H(F(z)). \tag{2}$$

Equation (1) is a special case of equation (2) for $H = \exp_b$. Then, multiplication is a superfunction of summation (addition of a constant), exponentiation is a superfunction of multiplication, and solution $F$ of equation (1) is a superfunction of the exponential, id est a super-exponential.

The special case of a super-exponential, holomorphic at least in the right hand side of the complex plane, is called «tetrational», $F = \mathrm{tet}_b$, if it satisfies the additional condition

$$F(0) = 1. \tag{3}$$

Four examples of $\mathrm{tet}_b$ are shown in figure 1 for $b = \sqrt{2}$, $b = \exp(1/\mathrm{e})$, $b = 2$ and $b = \mathrm{e}$.

The tetrational $\mathrm{tet}_b(z)$ can be interpreted as result of exponentiation applied to unity $z$ times, at least for integer values of $z$:

$$\mathrm{tet}_b(z) = \underbrace{\exp_b\Big(\exp_b\big(..\exp_b(1)..\big)\Big)}_{z \text{ exponentiations.}} \tag{4}$$

© 2010 Kouznetsov D.

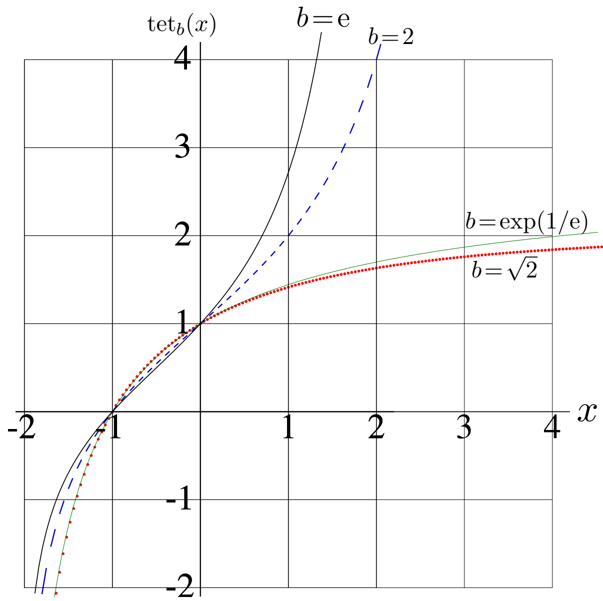


**Fig. 1.** Tetrational $\mathrm{tet}_b(x)$ at base $b = \mathrm{e}$ (thick solid), $b = 2$ (dashed), $b = \exp(1/\mathrm{e})$ (thin solid) and $b = \sqrt{2}$ (dotted) as holomorphic solutions of eqs. (1), (2), versus real $x$.

The name «tetrational» indicates, that this function is fourth in the sequence of functions (increment, addition, multiplication, exponential, tetrational, pentational, ...), where each element (except the zeroth element) is a superfunction for the previous element, and also the transfer function for the next element. The physical applications of the superfunctions, that justify the term «transfer function» are suggested in [8, 11, 12]; the superfunction and its inverse allow to evaluate the non-integer iteration of a function, in particular, such exotic functions as $\sqrt{\exp}$ by [1] and $\sqrt{!}$ by [11].

For complex values of the argument, the solution of equation (1) should be evaluated. The way of evaluation depends on $b$. At $1 < b \leqslant \exp(1/\mathrm{e})$, the regular iterations can be applied, recovering the function through its Schröder function [2, 3, 5–7, 9]; for larger values, the evaluation through the Cauchi integral [8, 10] is efficient. These representations were used to plot figure 1.

At base $b > \exp(1/\mathrm{e})$, the tetrational can be expressed through the contour integral [8], assuming, that it is holomorphic on the domain

$$C = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leqslant -2\}. \tag{5}$$

Such representation allows to express the derivative $\mathrm{tet}'$ and evaluate the inverse function, id est, arctetrational $\mathrm{ate} = \mathrm{tet}^{-1}$. Also, the name «superlogarithm», slog, is used for $\mathrm{tet}^{-1}$, although arctetrational is not a superfunction of the logarithm.

The arctetrational ate satisfies the equation

$$\mathrm{ate}\big(\exp(z)\big) = \mathrm{ate}(z) + 1. \tag{6}$$

The uniqueness of the function ate, biholomorphic on the domain

$$G = \{z \in \mathbb{C} : \ \mathrm{Re}(z) \geqslant \mathrm{Re}(L); \ |z| < |L|\} \tag{7}$$

was shown [8, 10]. Here $L \approx 0.318 + 1.337\,\mathrm{i}$ is a fixed point of logarithm, id est a solution of equation $L = \log(L)$. In the programming language Maple, this constant can be expressed as **conjugate(-LambertW(-1))**; while in notations of Mathematica it can be written as **Conjugate[-ProductLog[-1]]**.

The goal of this paper is to upgrade functions tet and ate to the status of special functions. For such upgrade, the precise and fast approximations should be supplied, allowing the efficient evaluation. Here, I consider the only one specific value $b = \mathrm{e}$; $\log = \ln = \log_{\mathrm{e}}$ and $H = \exp = \exp_{\mathrm{e}}$. However, tetrational for other values of base $b > \mathrm{e}^{1/\mathrm{e}}$ can be treated in a similar way. Below, I suggest the approximation for the function $\mathrm{tet} = \mathrm{tet}_{\mathrm{e}}$ shown in figure 1 with thick solid line; but this approximation is not limited to real values of the argument.

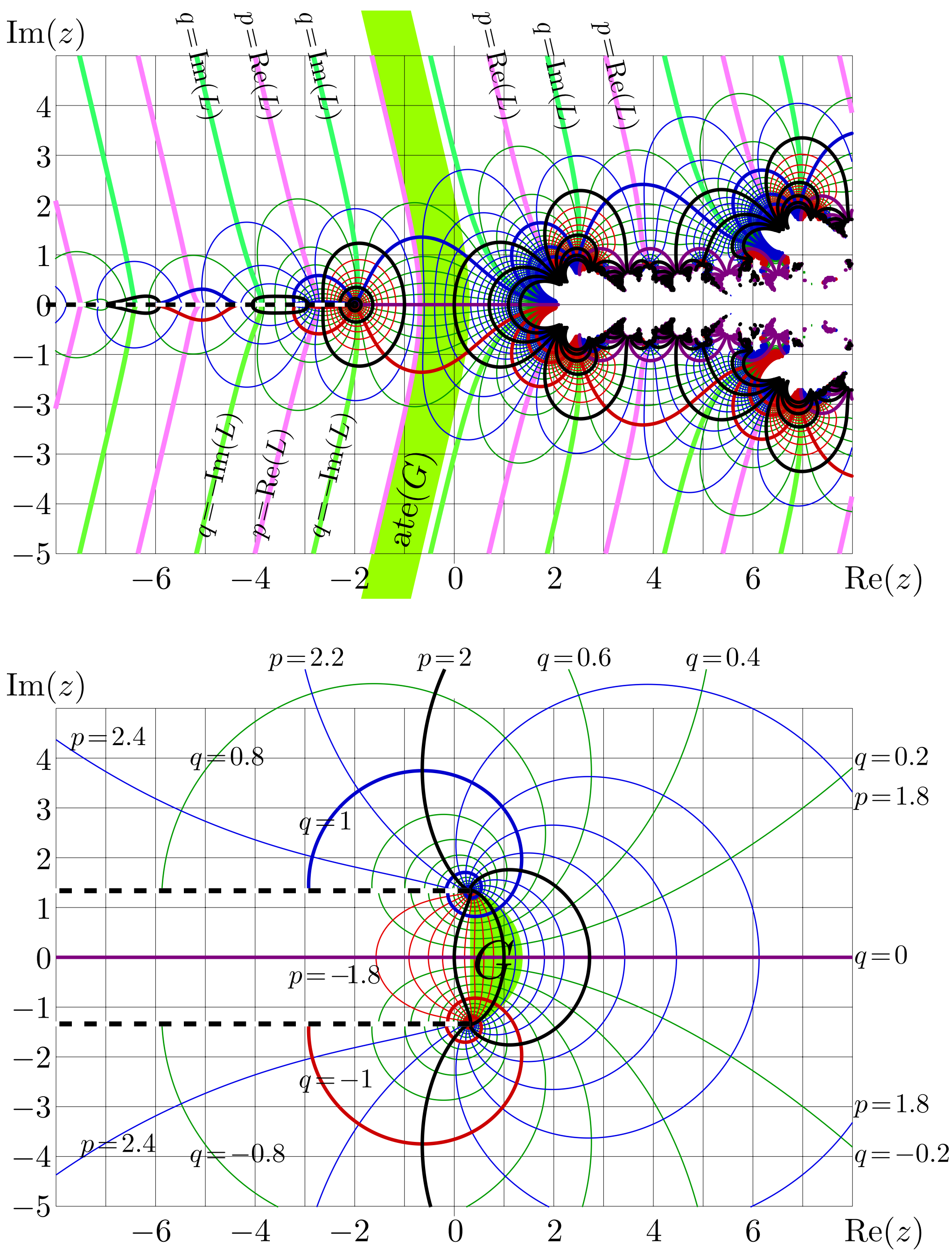


**Fig. 2.** Functions $f = \mathrm{tet}(z)$ and $f = \mathrm{ate}(z)$ in the complex $z$-plane. Lines show levels $p = \mathrm{Re}(f) = \mathrm{const}$ and $q = \mathrm{Im}(f) = \mathrm{const}$.

## 2. Properties of tet and ate

Any efficient approximation of any function should take into account its asymptotic properties. This section summarizes the basic knowledge about functions $f = \mathrm{tet}(z)$ and $f = \mathrm{ate}(z)$, that follow from the representation through the Cauchi integral [8]. Behavior of functions $f = \mathrm{tet}(z)$ and $f = \mathrm{ate}(z)$ in the complex plane is shown in figure 2 with lines $p = \mathrm{Re}(f) = \mathrm{const}$ and lines $q = \mathrm{Im}(f) = \mathrm{const}$. Levels of integer values $p$ and $q$ are shown with thick dark lines. Thick light lines indicate the levels $p{=}\mathrm{Re}(L)$ and $|q|{=}\mathrm{Im}(L)$. The thin lines indicate the intermediate levels. The shaded sickle indicates the set $G$ by (7). The upper tip of the sickle is $L$; the lower tip is $L^*$. The shaded strip shows the domain $\mathrm{ate}(G)$.

Function tet has the branch point $-2$. Position of the cut line, from $-2$ to $-\infty$ on the real axis, is determined by the condition $\mathrm{tet}(z^*) = \mathrm{tet}(z)^*$.

Function ate has two branch points, $L$ and $L^*$, and for the implementation we need to choose the cut lines. In the previous article [8], the cut lines run along the level $\mathrm{Re}(\mathrm{tet}(z)) = -2$ (see figure 8 in [8]); these cuts wind around the branch points, and the calculation of the cut line slows down the algorithm of the evaluation of function ate. Therefore, in this paper, the cutlines are placed horizontally.

Function tet asymptotically approaches its limiting values $L$ in the upper halfplane and $L^*$ in the lower halfplane. This approaching is seen in the figure 2 in the region, where the lines $p = \mathrm{Re}(L)$ look parallel to the lines $q = \mathrm{Im}(L)$. The approach to value $L$ is exponential [8]. The approximation of $\mathrm{tet}(z)$ at large values of $\mathrm{Im}(z)$ should use this property. In the left hand side of the complex plane and also in vicinity of the real axis function $f = \mathrm{tet}$ satisfies not only equation (1), but also the «inverse» equation, id est

$$\log\big(f(z+1)\big) = f(z) \quad \forall\, z \in C: \ |\mathrm{Im}(f(z))| < \pi. \tag{8}$$

Equations (1) and (8) simplify the fitting of the function. For the implementation of tetrational, it is sufficient to approximate it in some domain in the complex $z$-plane, that extends from $-\mathrm{i}\,\infty$ to $\mathrm{i}\,\infty$ in such a way that its overlap with set $\mathrm{Im}(z) = \mathrm{const}$ is not shorter than unity. This domain may partially overlap with the image $\mathrm{tet}(G)$ of domain $G$. In particular, such a domain can be $\mathrm{ate}(G)$, used in [10], or a strip $|\mathrm{Re}(z)| \leqslant 1/2$, used in [8]; this region can be also an «alternative strip» $-1 \leqslant \mathrm{Re}(z) \leqslant 0$, suggested for the independent verification of that result. In a similar way, for the implementation of function ate, it is sufficient to approximate it in some domain that extends from $L^*$ to $L$ is such a way that the exponential of the left margin belongs to the domain. The sickle $G$ gives an example of such a domain.

The approximations below are calculated using the discrete representation of the Cauchy contour integral [8] and extended to the whole complex plane using the properties of functions tet and ate.

## 3. Implementation of tetrational: fima

In order to distinguish functions tet and ate from their approximations, I give a specific name to each of them. The approximation of tet at large values of the imaginary part of its argument can be build up using the asymptotic representation

$$\mathrm{tet}(z) = L + \sum_{n,m} \mathcal{A}_{m,n} \exp\left(Lnz + \alpha m z\right). \tag{9}$$

The substitution into equation (1) and treating $\exp(Lz)$ as small parameter gives $\alpha = 2\pi\mathrm{i}$, and equations for the coefficients $\mathcal{A}$.

Taking into account only few of terms in expansion (9) gives the asymptotic approximation, I call it fima (Fast approximation at large IMaginary part of the Argument):

$$\mathrm{fima}(z) = \sum_{n=0}^{N} a_n \varepsilon^n + \beta\varepsilon \exp(2\pi \mathrm{i} z), \tag{10}$$

where $\beta$ is constant; the small parameter is

$$\varepsilon = \exp\big(Lz + R\big), \tag{11}$$

and the coefficients

$$a_0 = L \approx 0.3181315052047641353 + 1.3372357014306894089\,\mathrm{i}, \tag{12}$$

$$a_1 = 1, \tag{13}$$

$$a_2 = \frac{1/2}{L-1} \approx -0.1513148971556517359 - 0.2967488367322413067\,\mathrm{i}, \tag{14}$$

$$a_3 = \frac{a_2 + 1/6}{L^2-1} = \frac{2+L}{6(L-1)(L^2-1)} \approx -0.03697630940906762 + 0.09873054431149697\,\mathrm{i}, \tag{15}$$

$$a_4 = \frac{6+6L+5L^2+L^3}{24(L-1)^3(L+1)(L^2+L+1)} \approx 0.0258115979731401398 - 0.017386962126530755\,\mathrm{i}, \tag{16}$$

$$a_5 = \frac{24+36L+46L^2+40L^3+24L^4+9L^5+L^6}{120(L-1)^4(L+1)^2(1+L+2L^2+L^3+L^4)} \approx -0.0079444196 + 0.00057925018\,\mathrm{i}. \tag{17}$$

Parameter $R$ is introduced in order to set $a_1 = 1$ and keep simple expressions of other coefficients $a$ through the fixed point $L$ of the logarithm. Parameter $R$ can be defined as a complex number such that, at fixed values of $\mathrm{Re}(z)$ and large values $\mathrm{Im}(z) \gg 1$,

$$\mathrm{tet}(z) = L + \exp(Lz + R) + \mathcal{O}\big(\exp(2Lz)\big). \tag{18}$$

The increase of number of terms in the polynomial (10) and addition of polynomials with factors $\exp(2\pi \mathrm{i} z)$, $\exp(4\pi \mathrm{i} z)$, etc. improves the approximation, but for the prototype of the complex$\langle$double$\rangle$ numerical implementation, constructed below, it is sufficient to take 6 terms in the sum (10), setting $N = 5$, and only the single term proportional to $\exp(2\pi \mathrm{i} z)$.

I approximate parameters $R$ and $\beta$, fitting the numerical solution by [8]:

$$R \approx 1.0779614375280 - 0.94654096394782\ \mathrm{i}, \tag{19}$$

$$\beta \approx 0.12233176 - 0.02366108\ \mathrm{i}. \tag{20}$$

These values are expected to approximate the fundamental mathematical constants. Approximation fima by (10) is plotted in the top picture of the fig. 3 in the same notations as in fig. 2. The bottom picture shows the agreement function

$$D_0 = -\lg\Big|\exp\big(\mathrm{fima}(z-1)\big) - \mathrm{fima}(z)\Big|. \tag{21}$$

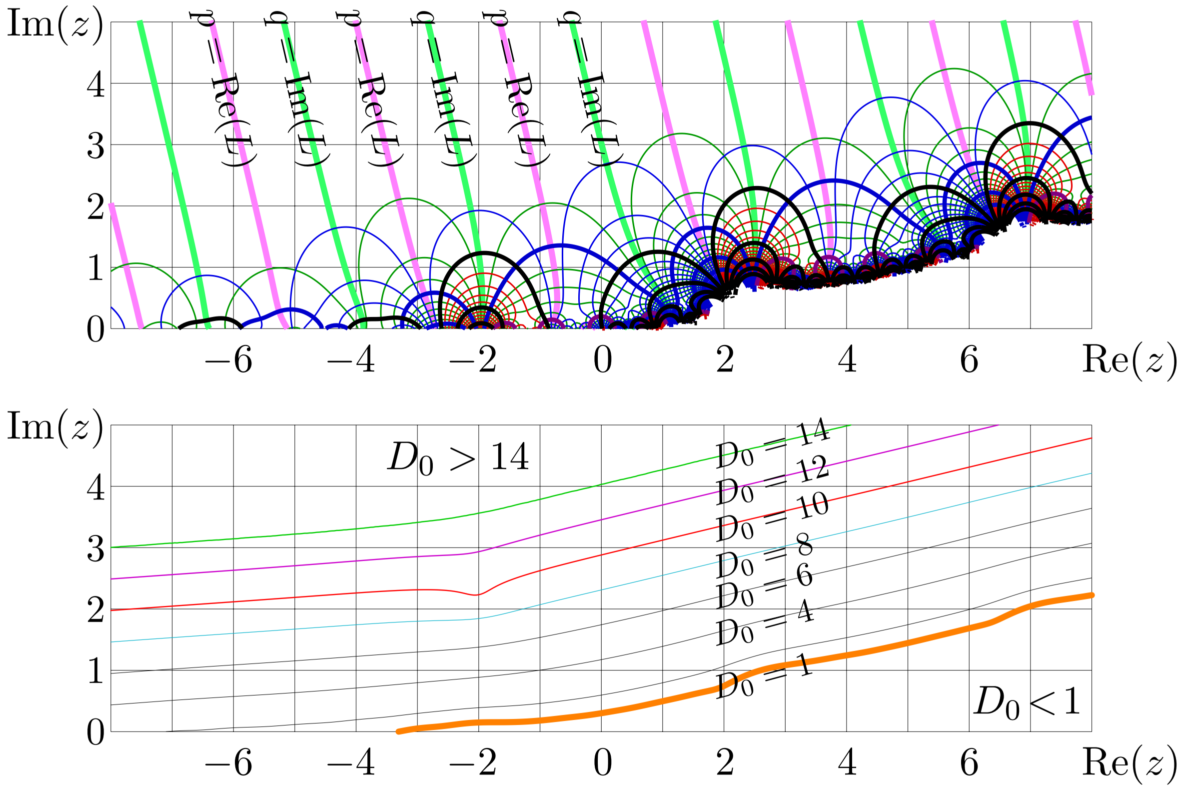


**Fig. 3.** Approximation $f = \mathrm{fima}(z)$ by (10) and agreement $D_0$ by (21) in the complex $z$-plane.

This function characterizes the residual at the substitution $F \to \mathrm{fima}$ into equation (1). The level $D = 1$ is shown with very thick light line at the bottom; the levels $D = 2, 4, 6, 8$ are shown with thin lines; The levels $D = 10, 12, 14$ are shown with thick lines. Roughly, the agreement function indicates, how many correct decimal digits may one get with this approximation. In particular, above the drawn lines, this approximation returns at least 14 significant figures, but for values below the lowest thickest line, even the first digit of this approximation is doubtful.

With the conjugated approximation $\mathrm{fima}(z^*)^*$, the fit by (10) covers a big part of the complex $z$-plane, but it is not good for small values of the imaginary part of the argument.

### 4. Approximation of tet: expansion at zero

The Taylor series has radius of convergence, equal to the distance from the expansion point to the nearest singularity. In the case of tetrational, the truncated sum of the MacLaurin series gives the approximation

$$\mathrm{naiv}(z) = \sum_{n=0}^{N-1} c_n z^n, \quad \mathrm{tet}(z) = \mathrm{naiv}(z) + \mathcal{O}(z^N) \tag{22}$$

usable at $|z| < 2$. The approximation with $N = 25$ is shown in the fig. 4; it is generated using the C++ source from CZ [13]. Approximations for the coefficients $c_n$ are shown in the first column of Table 1.

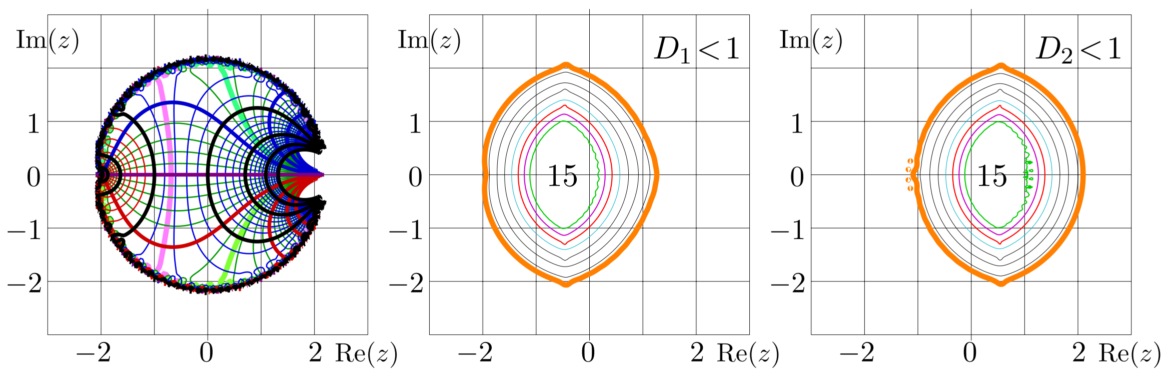


**Fig. 4.** Approximation $f = \mathrm{naiv}(z)$ of tet with the truncated Taylor expansion (22) at zero, the left picture; the agreements $D_1$ and $D_2$ by (23) and (24), the central and the right ones.

**Table 1**

**Coefficients of the series** (22)**,** (25) **and** (29)

| $n$ | $c_n$ | $s_n$ | $\mathrm{Re}(t_n)$ | $\mathrm{Im}(t_n)$ |
|---|---|---|---|---|
| 0 | 1.00000000000000 | 0.30685281944005 | 0.37090658903229 | 1.33682167078891 |
| 1 | 1.09176735125832 | 0.59176735125832 | 0.01830048268799 | 0.06961107694975 |
| 2 | 0.27148321290170 | 0.39648321290170 | −0.04222107960160 | 0.02429633404907 |
| 3 | 0.21245324817626 | 0.17078658150959 | −0.01585164381085 | −0.01478953595879 |
| 4 | 0.06954037613999 | 0.08516537613999 | 0.00264738081895 | −0.00657558130520 |
| 5 | 0.04429195209047 | 0.03804195209047 | 0.00182759574799 | −0.00025319516391 |
| 6 | 0.01473674209639 | 0.01734090876306 | 0.00036562994770 | 0.00028246515810 |
| 7 | 0.00866878181723 | 0.00755271038865 | 0.00002689538943 | 0.00014180498091 |
| 8 | 0.00279647939839 | 0.00328476064839 | −0.00003139436775 | 0.00003583704949 |
| 9 | 0.00161063129058 | 0.00139361740170 | −0.00001376358453 | −0.00000183512708 |
| 10 | 0.00048992723148 | 0.00058758348148 | −0.00000180290980 | −0.00000314787679 |
| 11 | 0.00028818107115 | 0.00024379186661 | 0.00000026398870 | −0.00000092613311 |
| 12 | 0.00008009461254 | 0.00010043966462 | 0.00000024961828 | −0.00000013664223 |
| 14 | 0.00001218379034 | 0.00001654344436 | 0.00000000637479 | 0.00000002270476 |
| 15 | 0.00000866553367 | 0.00000663102846 | −0.00000000341142 | 0.00000000512289 |
| 16 | 0.00000168778232 | 0.00000264145664 | −0.00000000162203 | 0.00000000031619 |
| 17 | 0.00000149325325 | 0.00000104446533 | −0.00000000038743 | −0.00000000027282 |
| 18 | 0.00000019876076 | 0.00000041068839 | −0.00000000001201 | −0.00000000013440 |
| 19 | 0.00000026086736 | 0.00000016048059 | 0.00000000002570 | −0.00000000002543 |
| 20 | 0.00000001470995 | 0.00000006239367 | 0.00000000000935 | 0.00000000000045 |
| 21 | 0.00000004683450 | 0.00000002412797 | 0.00000000000170 | 0.00000000000186 |
| 22 | −0.00000000154924 | 0.00000000928797 | −0.00000000000005 | 0.00000000000071 |
| 23 | 0.00000000874151 | 0.00000000355850 | −0.00000000000016 | 0.00000000000012 |
| 24 | −0.00000000112579 | 0.00000000135774 | −0.00000000000005 | −0.00000000000001 |
| 25 | 0.00000000170796 | 0.00000000051587 | −0.00000000000001 | −0.00000000000001 |

The 0th column of table 1 indicates the number $n$ of the coefficient; the first column indicates the value of the coefficient $c_n$ in equation (22).

The precision of the approximation (22) can be characterized with the agreement functions

$$D_1 = -\lg \Big| \exp(\mathrm{naiv}(z-1)) - \mathrm{naiv}(z) \Big|, \tag{23}$$

$$D_2 = -\lg \Big| \log(\mathrm{naiv}(z+1)) - \mathrm{naiv}(z) \Big|. \tag{24}$$

These functions are plotted in the central and the right hand side pictures in fig. 4. The digits «15» indicate the region, where the agreement is larger than 14. The figure indicates that at $|z| \leqslant 1$, the truncated Taylor series gives of order of 15 significant figures.

In order to extend the range of approximation, it worth to «switch out» the nearest logarithmic singularity at $-2$, expanding the function $\mathrm{tet}(z) - \log(z+2)$ instead of $\mathrm{tet}(z)$; let

$$\mathrm{maclo}(z) = \log(z+2) + \sum_{n=0}^{N-1} s_n z^n; \tag{25}$$

$$\mathrm{tet}(z) = \mathrm{maclo}(z) + \mathcal{O}(z^N). \tag{26}$$

The name maclo (MAClaurin expansion with LOgarithm) indicates, that the tetrational with substracted logarithm is approximated with the truncated MAClaurin series. The first coefficients $s$ of this expansion are evaluated in the second column in the table 1.

Function maclo is shown in the fig. 5 for $N = 101$. The generator [14] is used to plot the figure.

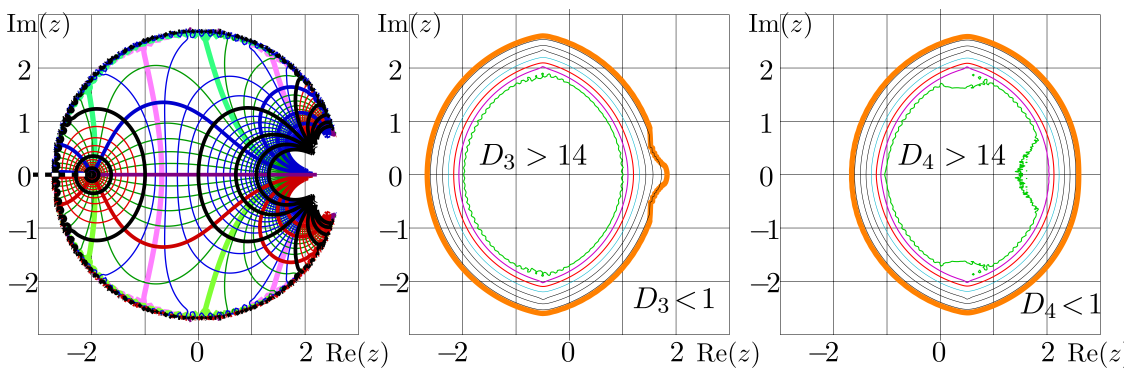


**Fig. 5.** Function $f = \mathrm{maclo}(z)$ by (25) at $N = 101$ in the complex $z$-plane, left; agreements $D_3$ and $D_4$ by (27) and (28), center and right.

The range of approximation of tet with function maclo is significantly wider, than that by the Taylor expansion of tet at zero. The right hand side of the same fig. 5, shows also the agreements

$$D_3 = -\lg\Big|\exp\big(\mathrm{maclo}(z{+}1)\big) - \mathrm{maclo}(z)\Big|\ , \tag{27}$$

$$D_4 = -\lg\Big|\log\big(\mathrm{maclo}(z{-}1)\big) - \mathrm{maclo}(z)\Big|\ . \tag{28}$$

Within the central loops, the residuals at the substitution $F \rightarrow \mathrm{maclo}$, $f \rightarrow \mathrm{maclo}$ into equations (1), (8) are of order of $10^{-15}$.

## 5. Approximation of tet : Taylor expansion at $3i$.

The plots of the agreement functions $D$ in fig. 3 and 5 indicate, that in the intermediate range $z \approx 3\mathrm{i}$, each of approximations $\mathrm{fima}(z)$ and $\mathrm{maclo}(z)$ return only few correct significant figures, if at al. For this reason I suggest the straightforward Taylor expansion at the intermediate point $z = 3\mathrm{i}$. I call it «tai» (TAylor expansion centered at at the Imaginary axis):

$$\mathrm{tai}(z) = \sum_{n=0}^{N-1} t_n \,(z - 3\mathrm{i})^n\,. \tag{29}$$

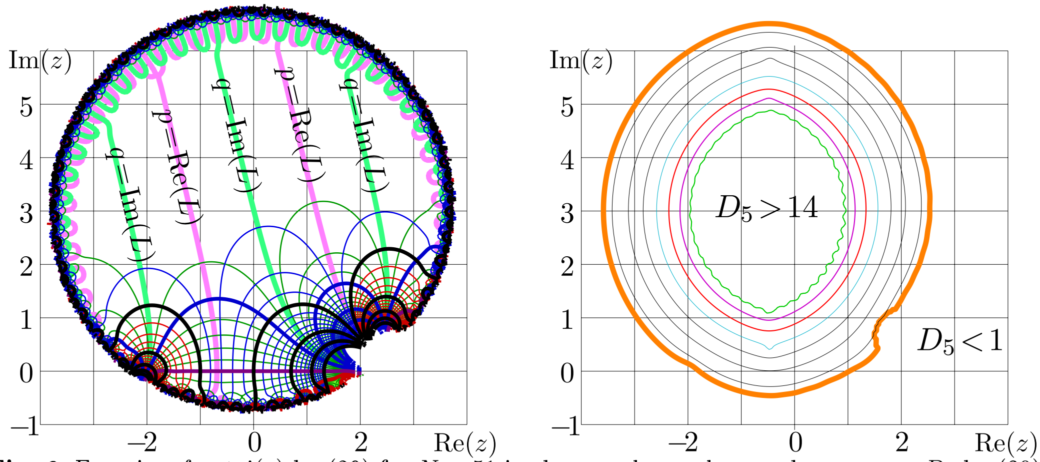


**Fig. 6.** Function $f = \mathrm{tai}(z)$ by (29) for $N = 51$ in the complex $z$-plane and agreement $D_5$ by (30).

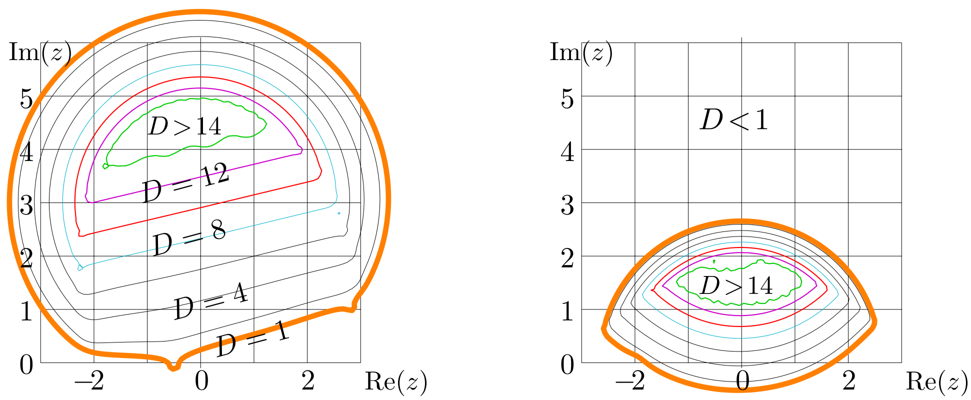


**Fig. 7.** Comparison of approximations tai by (29) to fima by (10), left, and to maclo by (25), right: agreements $D = D_6$ and $D = D_7$ by (31), (32) in the complex z-plane.

The coefficients of this series are evaluated in the last columns in the Table 1. For $N = 51$, function tai is shown in fig. 6.

The figure is plotted with generator [15]. The precision of the approximation of the solution of equations (1), (8) is characterized with agreement

$$D_5 = -\lg|\log(\mathrm{tai}(z+1)) - \mathrm{tai}(z)|. \tag{30}$$

This function is plotted in the right hand side of fig. 6.

The mutual agreement of the approximations above can be characterized with functions

$$D_6 = -\lg|\mathrm{fima}(z) - \mathrm{tai}(z)|, \tag{31}$$
$$D_7 = -\lg|\mathrm{maclo}(z) - \mathrm{tai}(z)|. \tag{32}$$

These functions are shown in fig. 7. Within the inner loops in the pictures of the fig. 7, the modulus of the difference between the approximations does not exceed $10^{-14}$. On the base of fig. 7, I suggest the following approximation:

$$\mathrm{fse}(z) = \begin{cases} \mathrm{fima}(z), & 4.5 < \mathrm{Im}(z), \\ \mathrm{tai}(z), & 1.5 < \mathrm{Im}(z) \leqslant \ 4.5, \\ \mathrm{maclo}(z), & -1.5 \leqslant \mathrm{Im}(z) \leqslant \ 1.5, \\ \mathrm{tai}(z^*)^*, & -4.5 \leqslant \mathrm{Im}(z) < -1.5, \\ \mathrm{fima}(z^*)^*, & \mathrm{Im}(z) < -4.5 \ . \end{cases} \tag{33}$$

This approximation can be compared to previous results. Below, I analyze the deviation $\mathrm{fse}(z) - F_4(z)$, where $F_4(z)$ is approximation, obtained by the straightforward implementation of the contour integral [8]. The left hand picture of fig. 8 shows the agreement

$$D_8 = -\lg \left|\mathrm{fse}(z) - F_4(z)\right| \tag{34}$$

of approximation fse with the approximation $F_4$ obtained through the direct implementation of the contour integral [8].

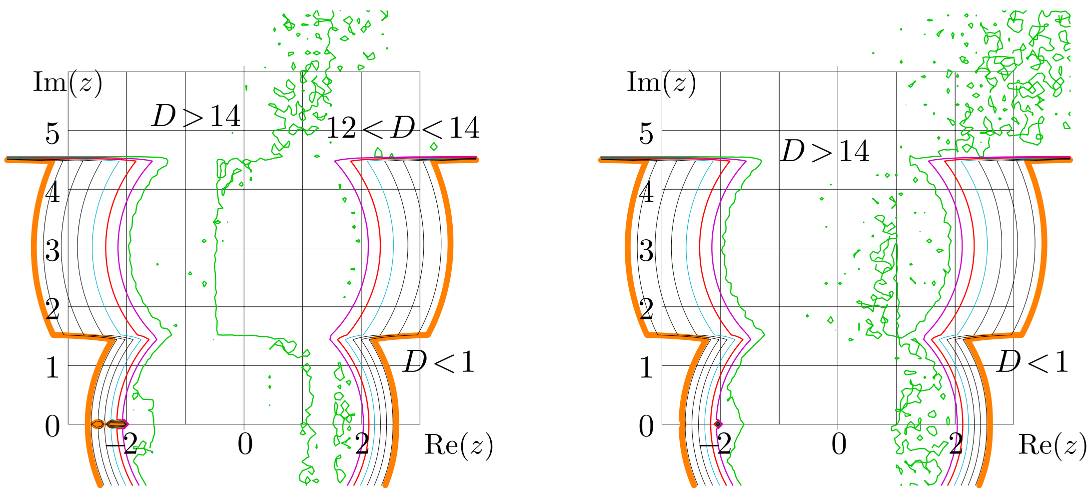


**Fig. 8.** Agreement $D = D_8$ by (34), left; the similar agreement for the contour integral with base domain shifted for $-0.5$ .

Fig. 8 reveals the defects of each approximation. The jumps at $\mathrm{Im}(z) = 1.5$ and at $\mathrm{Im}(z) = 2.5$ should be attributed to the transition from function maclo to function tai and from function tai to function fima in the combination fse. Jumps at half-integer values of $\mathrm{Re}(z)$ should be attributed to the discontinuities of function $F_4$, which extends the approximation with the contour integral, valid for $|\mathrm{Re}(z)| < 1$, from the interval $|\mathrm{Re}(z)| \leqslant 1/2$. The rounding errors appear as irregular dots. Within the strip $|\mathrm{Re}(z)| < 1.4$, the irregularities of all three approximations are of order of $10^{-14}$.

In the right hand side of fig. 8 the similar agreement is shown for function $F_5$, which is analogy of function $F_4$, but the base strip is displaced for $-1/2$. Approximation $F_5$ has jumps at integer values of the real part of the argument These jumps are also small; the agreement is at least not worse than that for function $F_4$.

In such a way, the deviation of all the approximations we count for today is of order of $10^{-14}$. On the base of fig. 7, 8, I suggest the final approximation FSE (Fast Super Exponential) of tetrational tet:

$$\mathrm{FSE}(z) = \begin{cases} \mathrm{FIMA}(z), & 4.5 < \mathrm{Im}(z), \\ \mathrm{TAI}(z), & 1.5 < \mathrm{Im}(z) \leqslant 4.5 \ , \\ \mathrm{MACLO}(z), & -1.5 \leqslant \mathrm{Im}(z) \leqslant 1.5 \ , \\ \mathrm{TAI}(z^*)^*, & -4.5 \leqslant \mathrm{Im}(z) < -1.5 \ , \\ \mathrm{FIMA}(z^*)^*, & \mathrm{Im}(z) < -4.5 \ , \end{cases} \tag{35}$$

where

$$\mathrm{FIMA}(z) = \begin{cases} \mathrm{fima}(z), & \mathrm{Im}(z) > 4 + 0.2379\,\mathrm{Re}(z), \\ \exp(\mathrm{FIMA}(z-1)), & \mathrm{Im}(z) \leqslant 4 + 0.2379\,\mathrm{Re}(z) \ , \end{cases} \tag{36}$$

$$\mathrm{TAI}(z) = \begin{cases} \mathrm{tai}(z), & |\mathrm{Re}(z)| \leqslant 0.5, \\ \log(\mathrm{TAI}(z+1)), & \mathrm{Re}(z) < -0.5, \\ \exp(\mathrm{TAI}(z-1)), & \mathrm{Re}(z) > 0.5 \ , \end{cases} \tag{37}$$

$$\mathrm{MACLO}(z) = \begin{cases} \mathrm{tai}(z), & |\mathrm{Re}(z)| \leqslant 0.5 \ , \\ \log(\mathrm{MACLO}(z+1)), & \mathrm{Re}(z) < -0.5 \ , \\ \exp(\mathrm{MACLO}(z-1)), & \mathrm{Re}(z) > 0.5 \ . \end{cases} \tag{38}$$

This approximation provides of order of 14 correct significant figures of the holomorphic tetrational tet and agrees with the previous results [8].

Up to my knowledge, the function FSE is the most precise and the fastest among ever reported approximations of the tetrational. Many terms are kept in the approximations (29) and (25) in order to provide the wide range of the overlapping in fig. 7 and 8. At the final step of the implementation, the number of terms can be reduced, boosting the algorithm without loss of the precision. Especially this applies to the evaluation of tetrational along the real axis: it is sufficient to get a good approximation of $\mathrm{tet}(z)$ for $|z| \leqslant 1/2$, which is only a quarter of the radius of the precise approximation with function maclo. By requests of colleagues the algorithm is translated from language C++ into language Mathematica [16]. As verification of this algorithm, the first, second, third and fourth derivatives of tetrational tet are plotted in fig. 9 as functions of real argument; however, the algorithm evaluates also the tetrational and its derivatives of complex argument.

The good overlapping of the ranges of approximation of tetrational tet by various algorithms confirm their validity. Perhaps, some increase of the relative error may take place at the sequential application of exp required at large values of the real part of the argument. At moderate values of the real part of the argument, all the 3 approximations FSL by (35), and $F_4$ by [8] and its modigication $F_5$ (see fig. 8) seem to have comparable errors at the level of $10^{-14}$.

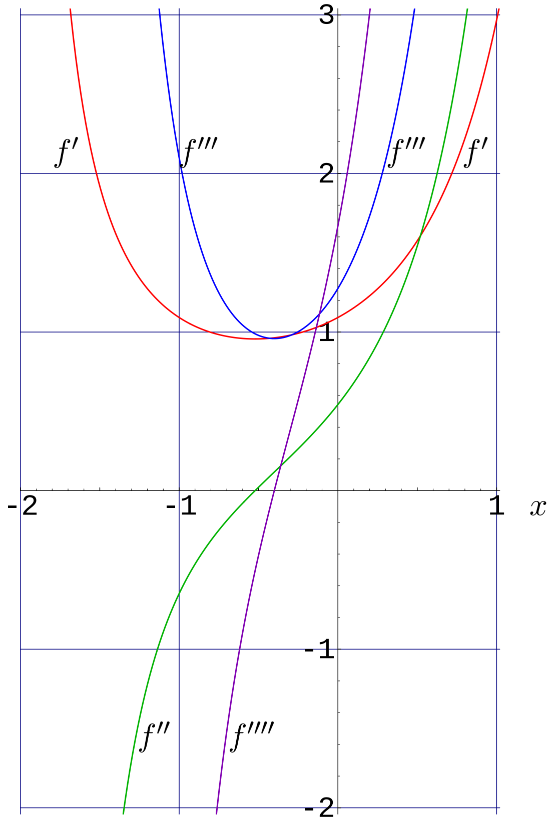


**Fig. 9.** First four derivatives of $f = \mathrm{tet}(x)$ as functions of real $x$.

## 6. Implementation of arctetrational: function FSL

The inverse function of tetrational, id est, arctetrational $\mathrm{ate} = \mathrm{tet}^{-1}$, satisfies the equations

$$\mathrm{ate}(z) \quad = \quad \mathrm{ate}(\exp(z)) - 1, \tag{39}$$

$$\mathrm{ate}(z) \quad = \quad \mathrm{ate}(\log(z)) + 1 \tag{40}$$

at least for $z \in G$; and $\mathrm{ate}(1) = 0$. In principle, function ate can be implemented as numerical solution of equation $\mathrm{tet}(\mathrm{ate}(z)) = z$; however, such implementation is much slower than the approximation with the appropriate elementary functions.

The first (and naive) attempt to approximate function ate is, of course, the Taylor expansion at unity. The coefficients of such an expansion can be found, inverting the powerseries naiv by (22). The radius of convergence of the resulting expansion is $|L| \approx 1.5$; and the approximation is especially poor in vicinity of the fixed points $L$ and $L^*$ of logarithm.

The better approach is to expand the function

$$\mathrm{ate}(z) - \frac{\log(z-L)}{L} - \frac{\log(z-L^*)}{L^*} \tag{41}$$

at $z = 1$. Such an expansion leads to the approximation

$$\mathrm{fsl}(z) \quad = \quad \frac{\log(z-L)}{L} + \frac{\log(z-L^*)}{L^*} + \sum_{n=0}^{N-1} u_n (z-1)^n; \tag{42}$$

$$\mathrm{ate}(z) \quad = \quad \mathrm{fsl}(z) + \mathcal{O}\big((z-1)^N\big) \tag{43}$$

The approximations of first coefficients of such an expansion are shown in table 2. The approximation fsl at $N=70$ is shown in fig. 10.

**Table 2**

**Coefficients $u_n$ in expansion** (42)

| $n$ | $u_n$ | $n$ | $u_n$ | $n$ | $u_n$ |
|---|---|---|---|---|---|
| 0 | 1.41922521550451 | 10 | 0.00000003111805 | 20 | 0.00000000002293 |
| 1 | −0.02606629029752 | 11 | 0.00000002940887 | 21 | −0.00000000002462 |
| 2 | 0.00173304781808 | 12 | −0.00000001896929 | 22 | 0.00000000000666 |
| 3 | −0.00001952130725 | 13 | 0.00000000351784 | 23 | 0.00000000000322 |
| 4 | −0.00006307006450 | 14 | 0.00000000204270 | 24 | −0.00000000000354 |
| 5 | 0.00002567895998 | 15 | −0.00000000171995 | 25 | 0.00000000000096 |
| 6 | −0.00000559010027 | 16 | 0.00000000039882 | 26 | 0.00000000000051 |
| 7 | −0.00000007279712 | 17 | 0.00000000019328 | 27 | −0.00000000000055 |
| 8 | 0.00000065148872 | 18 | −0.00000000019113 | 28 | 0.00000000000014 |
| 9 | −0.00000027698138 | 19 | 0.00000000004947 | 29 | 0.00000000000009 |

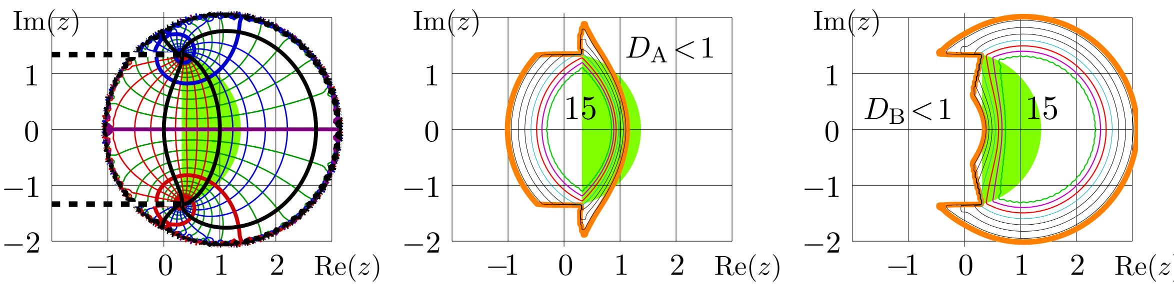


**Fig. 10.** Approximation fsl$(z)$ by (42), left, and the agreements by (44) and (45); the domain $G$ by (7) is shaded.

Formally, the Taylor series of function (41) developed at $z = 1$ has the same radius of convergence as the direct Taylor expansion of function ate. However, practically, at the numerical implementation, the convergence for representation (42) is much faster, than that for the straightforward Taylor expansion. Function fsl approximates function ate even at the edge of the range of convergence, and, in particular, in vicinity of the tips of the sickle $G$ by (7), id est, in vicinity of points $L$ and $L^*$. Function fsl has the same branchpoints $L$ and $L^*$, as function ate, and also is infinite in these points.

In order to characterize the residuals at the substitution ate $\rightarrow$ fsl into equations (39), (40), the agreements

$$D_{\mathrm{A}} \quad = \quad -\lg\Big|\mathrm{fsl}\big(\exp(z)\big) - 1 - \mathrm{fsl}(z)\Big|, \tag{44}$$

$$D_{\mathrm{B}} \quad = \quad -\lg\Big|\mathrm{fsl}\big(\log(z)\big) + 1 - \mathrm{fsl}(z)\Big| \tag{45}$$

are plotted in the central and right hand side pictures of fig. 10. As before the symbol «15» indicates the region, where the agreement is better than 14. (Within the inner loops, the residuals are smaller that $10^{-14}$.)

The range of approximation can be extended with function

$$\mathrm{FSE}(z)=\begin{cases}\mathrm{fsl}(z), & |\mathrm{Im}(z)|<\mathrm{Im}(L)\ \& \\ & |z-1|\leqslant|\log(z)-1|\ \&\ \ |z-1|\leqslant|\exp(z)-1|,\\ \mathrm{FSL}(\exp(z))-1, & |\mathrm{Im}(z)|<\mathrm{Im}(L)\ \&\ |z-1|>|\exp(z)-1|,\\ \mathrm{FSL}(\log(z))+1, & |\mathrm{Im}(z)|\geqslant\mathrm{Im}(L)\ \text{or}\ \ |z-1|>|\log(z)-1|.\end{cases} \tag{46}$$

Through the extension (46), function fsl allows to cover the whole complex plane with the single approximation with the elementary function by (42). In order to check the mutual consistency or approximations FSE and FSL, consider the agreements

$$D_{\mathrm{c}} = -\lg\Big|\mathrm{FSL}\big(\mathrm{FSE}(z)\big)-z\Big|, \tag{47}$$

$$D_{\mathrm{d}} = -\lg\Big|\mathrm{FSE}\big(\mathrm{FSL}(z)\big)-z\Big|. \tag{48}$$

These agreements are shown in fig. 11.

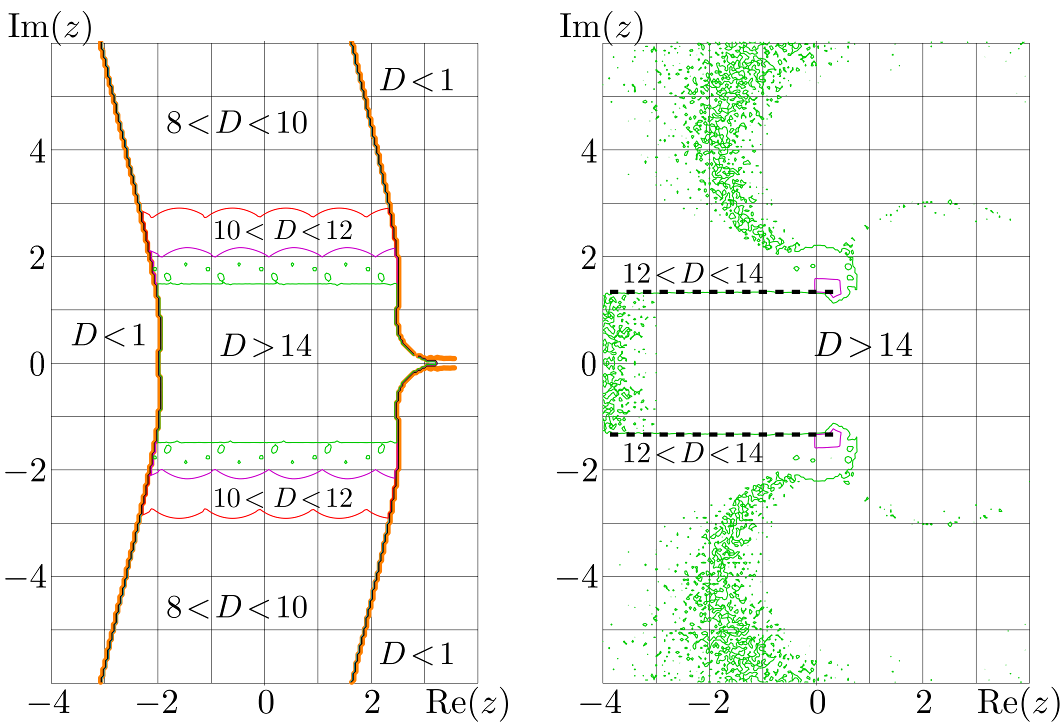


**Fig. 11.** Agreements of approximations FSE and FSL by (47), left and (48), right.

The figure confirms the good precision of the approximations. The error of these approximations is comparable to the rounding errors at the complex⟨double⟩ variables. The algorithms suggested are robust.

## Conclusion

The numerical algorithm FSE by equation (35) approximates the holomorphic tetrational (super-exponential) on base e. The algorithm FSL by (46) approximates the inverse function. Up to date, these are the fastest and most precise algorithms. These algorithms can be prototypes for the numerical implementations of tetrational and arctetrational in compilers of next generation.

**Acknowledgment.** The author is grateful to Henryk Trappmann for his help and critics.

## References


1. *Kneser H.* Reelle analytische Lösungen der Gleichung $\varphi(\varphi(x)) = \mathrm{e}^x$ und verwandter Funktionalgleichungen // J. für die reine und angewandte Math.—1950.—Vol. 187.—P. 56–67.
2. *Szekeres G.* Regular iteration of real and complex functions // Acta Math.—1958.—Vol. 100.—P. 203–258.
3. *Szekeres G.* Fractional iteration of exponentially growing function // J. Austral. Math. Soc.—1961.—Vol. 2.—P. 301–320.
4. *Knoebel R. A.* Exponential reiterated // Amer. Math. Monthly.—1981.—Vol. 88.—P. 235–252.
5. *Bromer N.* Superexponentiation // Math. Mag.—1987.—Vol. 60, № 3.—P. 169–174.
6. *Walker P.* Infinitely differentiable generalized logarithmic and exponential functions // Math. of Comp.—1991.—Vol. 196.—P. 723–733.
7. *Walker P.* On the solutions of an Abelian functional equation // J. Math. Anal. and Appl.—1991—Vol. 155, № 1.—P. 93–110.
8. *Kouznetsov D.* Solution of $F(z{+}1) = \exp(F(z))$ in complex $z$-plane // Math. of Comp.—2009.—Vol. 78.—P. 1647–1670.
9. *Kouznetsov D., Trappmann H.* Portrait of the four regular super-exponentials to base sqrt(2) // Math. of Computation.—2010.— Vol.79,—P. 1727–1756.
10. *Trappmann H., Kouznetsov D.* Uniqueness of analytic Abel functions in absence of a real fixed point.—2010.—(to appear). Preprint: URL: `http://www.ils.uec.ac.jp/~dima/PAPERS/2009uniabel.pdf`
11. *Kouznetsov D., Trappmann H.* Superfunctions and square root of factorial // Moscow Univ. Physics Bulletin. —2010.—Vol. 65, № 1.—P. 6–12.
12. *Kouznetsov D.* Holomorphic extension of the logistic sequence // Moscow Univ. Physics Bulletin.—2010.—Vol. 65, № 2.—P. 91–98. URL: `http://www.springerlink.com/content/u712vtp4122544x4`
13. Code for fig. 4 URL: `http://en.citizendium.org/wiki/TetrationPolynomial25power.jpg/code`
14. Code for fig. 5 URL: `http://en.citizendium.org/wiki/TetrationApproLP100.jpg/code`
15. Code for fig. 6 URL: `http://en.citizendium.org/wiki/TetrationTailorExpansion3ipower25.jpg/code`
16. Code for fig. 9 URL: `http://en.citizendium.org/wiki/TetrationDerivativesReal.jpg/code`
17. Code for fig.10 URL: `http://en.citizendium.org/wiki/SLOGappro50.jpg/code`


*Received February 16, 2009*

Kouznetsov Dmitrii
Institute for Laser Science,
University of Electro Communications,
*researcher*
1-5-1 Chofugaoka, Chofushi, Tokyo, 182-8585, Japan
E-mail: `dima(at)ils.uec.ac.jp`

## ТЕТРАЦИЯ КАК СПЕЦИАЛЬНАЯ ФУНКЦИЯ

Кузнецов Д. Ю.

Голоморфная тетрация (суперэкспонента) по основанию $e$ и ее обратная функция (арктетрация) аппроксимированы элементарными функциями.

**Key words:** тетрация, суперфункция, функция Абеля, голоморфная функция, аналитичная функция, суперэкспонента, суперлогарифм, аккуратная аппроксимация функций, специальные функции, итерирование функций, нецелые итерации.