Superfunctions for optical amplifiers

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The realistic transfer function T of the homogeneously-pumped laser amplifier is expressed in terms of the LambertW function. The intensity F inside the amplifier is reconstructed as superfunction of T by the method of regular iteration. The reconstructed function is compared to the analytical solution of the transfer equation. The method of regular iteration is suggested as a tool for characterization of laser materials from the measurement of the transfer function of a bulk sample.

Keywords: Theoretical Tool, laser materials, transfer function, superfunction

1. Introduction

Precision of characterization of laser materials usually refers to two significant figures, and is very far from the limits due to inhomogeneity of the space-time, that would allow to have of order of 20 significant figures. One of reasons for this is, that the laser materials are no so identical, as atoms and simple molecules are. Another reason is the problem of measurement of the gain and absorption of the optically-thin materials, while the intensity does not change so much and the property can be attributed to the certain intensity.

For example, until now, it is difficult to estimate, with how many significant figures does hold the identity for the quasi-two level medium,

$$G(I_{\rm pump}, I_{\rm signal})/G_0 + A(I_{\rm pump}, I_{\rm signal})/A_0 = 1 \qquad (1)$$

where $G(I_{\rm pump}, I_{\rm signal})$ is gain at the lasing frequency, assuming given intensities $I_{\rm pump}$, $I_{\rm signal}$ at the pump frequency and at the signal frequency; , $A(I_{\rm pump}, I_{\rm signal})$ is absorption at the pump frequency, as function of the same arguments; $G_0 =$ const is gain at the strong pump and $A_0 =$ const is absorption of pump at the strong signal.[1)] In order to make measurements of gain at a given intensity, the sample should be optically thin. But the precise measurements of small variation of intensity in the optically-thin sample are difficult. For the optically-thick sample, it is possible to get many decimal digits in the measurement of the mean gain, but it is difficult to guess, namely which intensity does this gain correspond to.

For the testing of the specific model, the methodologically-correct would be measurement of the transfer function of a bulk (id est, optically thick sample), characterising $I_{\rm out} = T(I_{\rm in})$, where $I_{\rm in}$ and $I_{\rm out}$ are input and output intensities (either pump, or signal), and T is corre-

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sponding transfer function. Then, this transfer function can be compared with the simulations of propagation of light in the nonlinear medium, giving an estimate for the precision of the model.

For the new laser media, with unknown kinetics, the simulation of the whole experiment, taking into account the finite thickness of the sample, may be difficult; and it would be desirable to have a tool for the measurement of gain and measurement of absorption of pump at the given intensities I_{pump} and I_{signal}. In the measurement of the gain, in principle, it is possible to keep the pump constant (for example, at the lateral delivery of pump), but variation of signal should be significant in order to measure it with many decimal digits. The precise measurements of teh transfer function T are possible, but the variation of the signal inside the sample should be somehow recovered. Similar ideology can be applied to the absorption of pump, which depends on the pump intensity.

Let F be intensity of the pump or that of the signal in the continuouswave experiment. This F can be also interpreted as a fluence in the experiment with a short pulse, when the dependence of gain on fluence is analyzed; for the fluences, the general formulas below remain the same. For simplicity, assume, that F is intensity of light in a medium.

Let the transfer function T of some sample be precisely measured, expressing the output as a function of the input. Is it possible to reconstruct the evolution of intensity $F(x)$ along the coordinate x of the propagation?

For simplicity, assume, that the coordinate x is measured in units of thickness of the sample (which is supposed to be uniformly pumped for the propagation of the signal). Then the question above refers to the transfer equation

$$F(x+1) = T(F(x)) \tag{2}$$

Equation (2) could corresponds to a long medium; but if at some coordinate x intensity is $F(x)$, then, after to pass the thickness of the sample, the intensity becomes $F(x+1)$, and it is also $T(F(x))$.

For the given transfer function T, the solution F is called "superfunction". With superfunction, the iteration of the transfer function T can be vritten as follows:

$$T^n(z) = \underbrace{T\Big(T\big(...T(z)...\big)\Big)}_{z \text{ evaluations of function } T} = F\Big(n + F^{-1}(z)\Big) \tag{3}$$

where F^{-1} is minus-first iteration of function F, id est, inverse function of F, so, that $F(F^{-1}(z)) = z$. In expression (3), the number n of iterations has no need to be integer; in particular, the half-iteration of factorial and that of exponential can be evaluated[6,7] in such a way. Corresponding half-iterations can be denoted with $\sqrt{!}$ and $\sqrt{\exp}$; however, $\sqrt{\exp}(z) = \exp^{1/2}(z)$ should not be confused with $\sqrt{\exp(z)} = \exp(z/2)$. Upper superscript after the name of the function denotes either the derivative (it it is prime) or number of iterations of this function. In these notations, $\sin^2(x) = \sin(\sin(x))$ and never $\sin(x)^2$.

The solution of the transfer equation (2) is not unique, even if we measure the intensity in the units of intensity of saturation and choose the origin of system of coordinates in such a way, that $F(0) = 1$. If F is solution, then another solution G can be constructed as

$$G(x) = F(x + \eta(x)) \tag{4}$$

where η is real periodic function with period unity. For this reasons the equation (2) had been qualified as "hopeless"; the reconstruction of the physically-correct superfunction F from the transfer function T was qualified as impossible an.[8–10]

The superfunction F becomes unique at the specific additional conditions on the behavior in the compelex plane,[7,14,15] and the method of regular iterations (if applicable) returns namely this solution. This has been verified for the quadratic transfer function,[14] for T = Factorial, for $T = \exp_b$ at $1 < b < \exp^2(-1)$.[15] In principle, these transfer functions may have applications in various branches of physics, but it is difficult to realize, for example, the amplifier with factorial transfer function. Realistic transfer functions of optical amplifiers no not raise their derivatives at large values of the argument, but, contrary, show saturation, describing reduction of the amplification coefficient for the strong input signal.

In this paper, the realistic transfer function $T(z) = \mathrm{Doya}(z) = \mathrm{LambertW}(z\mathrm{e}^{1+z})$ is considered. For this case, the superfunction can be expressed analytically, $F(x) = \mathrm{Tania}(x-1) = \mathrm{LambertW}(\mathrm{e}^x)$. The efficient algorithms for evaluation of functions Tania and Doya are available, their properties are known[11,12] and they can be considered as special functions. The approximations of F are constructed with method of regular iterations and compared to the exact solution.

The goal of this paper to convince colleagues to perform the precise measurements of the transfer functions for uniformly pumped bulk samples. The the precision of the reconstruction of the properties of the material with formalism of superfunctions should be compared to the precision of the direct measurements with optically thin samples. This formalism should help in the correct treatment of experimental data for the optically thick samples, at least to avoid publications of results contradicting the Second Law of thermodynamics.[13]

2. Method of regular iteration

In this section, the method of regular iterations[7,14,15] is repeated for the case, when zero is fixed point of the transfer function, id est, $T(0) = 0$. This neglects the amplified spontaneous emission, the zero input leads to the zero output.

Search the approximation $\tilde{F}$ of the solution F of the transfer equation (2) as expansion with exponentisps

$$\tilde{F}(x) = \varepsilon + a_2\,\varepsilon^2 + a_3\,\varepsilon^3 + .. \tag{5}$$

where $\varepsilon = \mathrm{e}^{kx}$; k and a_2, a_3,.. are constant coefficients; parameter e^k has sense of the coefficient of amplification a weak signal. ε is supposed to be small; then, the truncation of series in the expansion (5) give the primary approximation $\tilde F$. Parameters k and a are easy to evaluate.

For function with displaced argument in the left hand side of the transfer equation (2), we have expression

$$\tilde F(x+1) = \mathrm{e}^k \varepsilon + a_2\, \mathrm{e}^{2k}\, \varepsilon^2 + a_3\, \mathrm{e}^{3k}\, \varepsilon^3 + .. \tag{6}$$

The expansion of the right hand side of the transfer equation (2), at the substitution of $\tilde F$ instead of F, can be written as follows:

$$T(\tilde F(x)) = T'\varepsilon + T' a_2 \varepsilon^2 + T' a_3 \varepsilon^3 + .. + \frac{T''}{2}(\varepsilon + a_2\varepsilon^2 + ..)^2 + \frac{T'''}{6}(\varepsilon + ..)^3 + .. \tag{7}$$

where $T' = T'(0)$, $T'' = T''(0)$, $T''' = T'''(0)$ are derivatives of the transfer function T at zero. Then, from the transfer equation (2) we have relations

$$\mathrm{e}^k = T' \tag{8}$$

$$\mathrm{e}^{2k} a_2 = T' a_2 + T''/2 \tag{9}$$

$$\mathrm{e}^{3k} a_3 = T' a_3 + T'' a_2 + T'''/6 \tag{10}$$

and so on, determining parameters in the expansion (5); in particular,

$$k = \ln(T') \tag{11}$$

$$a_2 = \frac{T''/2}{(T'-1)T'} \tag{12}$$

$$a_3 = \frac{T'' a_2 + T'''/6}{((T')^2-1)T'} \tag{13}$$

The truncated series with only one or few terms in expansion (5) gives good approximation $\tilde F$ for small values of ε. At positive k, the approximation refers to the large negative x. For moderate values of x, the transfer function can be approximated with

$$F(x) \approx T^n\Big(\tilde F(x-n)\Big) \tag{14}$$

determining the regular iteration of the transfer function as a way to approximate the super-function F with any precision required, id est, to evaluate it.

In previous works,[7,14,15] for various transfer functions, using just "double" arithmetics, of order of 14 correct decimal digits were achieved, evaluating the superfunctions with regular iteration, similar to that described above. This indicates the good stability of the algorithm, and such a precision seems to exceed the needs of characterization of the laser materials of 21th century. In order to confirm, that this method works also for the transfer functions with saturation, typical for the laser media, in the next section such an example is considered.

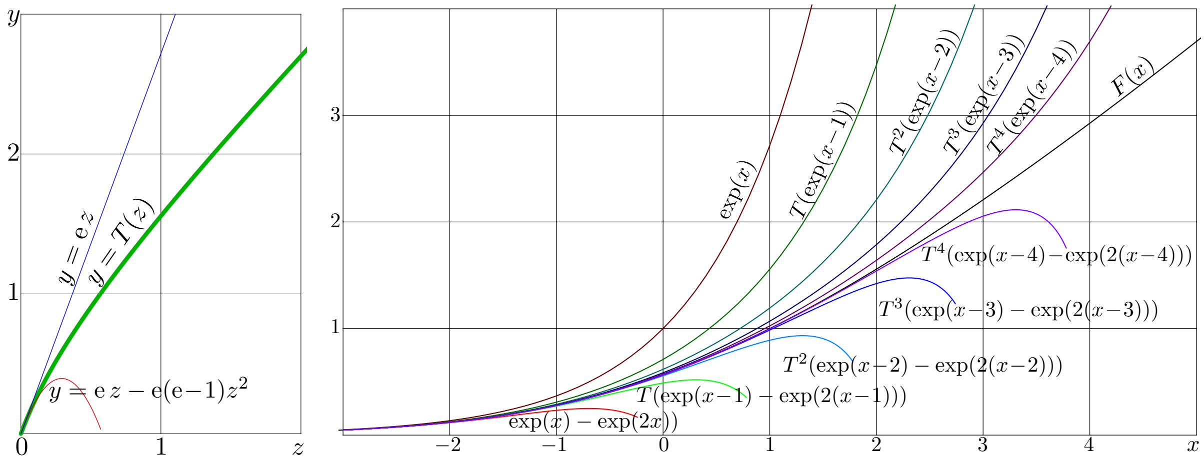

Fig. 1. Left: Transfer function T by (15) and its approximations. Right: superfunction F and its approximations $\tilde F$ by the regular iteration (14) for n=0..4.

3. The Doya function and the Tania function

The simple transfer function for the laser amplifier can be expressed through the LamebrtW function; let

$$T(x) = \mathrm{Doya}^1(x) = \mathrm{LambertW}(x\,\mathrm{e}^{1+x}) \tag{15}$$

Graphic of this function T is shown in the left hand side of Figure 1 with thick line. Properties of function Doya are described[11)] and the efficient (id est, fast and precise) algorithm for the evaluation is supplied. At small values of the argument,

$$T(z) = \mathrm{e}\,z - \mathrm{e}(\mathrm{e}-1)\,z^2 + O(z)^3 \tag{16}$$

The corresponding linear and quadratic approximations of T are shown in the left hand side of Figure 1 with thin lines. Evaluations by (11) gives $k = 1$, so, $\varepsilon = \mathrm{e}^x$; evaluation by (12) gives $a_2 = -1$. The primary approximation by the expansion (5) with single term gives $\tilde F(x) = \exp(x)$; that with two terms give $\tilde F(x) = \mathrm{e}^x - \mathrm{e}^{2x}$. These approximations are shown with uppest and lowest curves in the right hand side of figure 1. The regular iteration by (14) of these functions are plotted for $n = 1, 2, 3, 4$.

For this example, the superfunction F can be expressed analytically through the Tania function

$$F(x) = \mathrm{Tania}(x-1) = \mathrm{LambertW}(\mathrm{e}^x) \tag{17}$$

Properties of the Tania function are described[12)] and the efficient algorithm for the evaluation is supplied. This exact solution is also plotted in the same figure.

The same function F can be obtained as solution of differential equation

$$F'(x) = \frac{F(x)}{1 + F(x)} \tag{18}$$

that corresponds to the amplification of light in the gain medium with simple kinetics. In such a way, Figure 1 shows that the regular iteration quickly converges to the physically-meaningful solution.

One may expect the same method to return the physically meaningful solution, id est, distribution of the intensity of the signal in the uniformly pumped amplifier, also in more complicated cases, while the simple analytic representation for the transfer function and that for the superfunction are not yet available. This can be formulated as the following conjecture:

For realistic transfer function T, the method of regular iterations by (6)-(14) gives namely that solution of the transfer equation (2), that corresponds to the distribution of intensity F in the homogeneously pumped amplifier.

4. Conclusions

Superfunctions provide tool for the characterization of laser materials, allowing the recovery of the properties from the measurement of the transfer function T of the optically-thick uniformly pumped samples. This recovery implies the construction of the appropriate superfunction F, that has sense of distribution of intensity (or fluence) along the amplification. For realistic transfer function T by (15), the method of regular iteration (14) chooses the physically-meaningful function F by (17) among variety of solutions of the transfer equation (2) .

References

1) D. Kouznetsov, J.-F. Bisson, K. Takaichi, and K. Ueda: J. Opt. Soc. Am. B, **22**, 1605.

2) K. Ueda, H. Sekiguchi, Y. Matsuoka, H. Miyajima, and H. Kan. I CLEO?Pacific Rim99/ 217 (1999)

3) Hamamatsu Photonics K.K. Laser group: The Fiber Disk Laser explained. Nature Photonics sample, p14-15 (2006)

4) P. Kano, D. Kouznetsov, J. V. Moloney, and M. Brio: IEEE J.Quantum Electron. **40**, (2004), No.9, 1301

5) D. Kouznetsov, and J. V. Moloney: IEEE JQuantum Electron. **40**, (2004), Issue 4, 378.

6) H.Kneser: Journal fur die Reine und Angewandte Mathematik 187 (1950): 56.

7) D. Kouznetsov and H. Trappmann: Moscow University Physics Bulletin, **65** (2010), No.1, 6

8) H. Yoneda. Recovery of intensity inside the amplifier from its transfer function is impossible. Private communication, 2010.

9) A. Shirakawa. It is not possible to reconstruct the intensity inside the amplifier just from its transfer function. Private communication, 2011.

10) A. Kaminsky. There is no sense to analyze the transfer function of an amplifier without to assume the specific model for the gain medium. Private communication, 2011.

11) `http://tori.ils.uec.ac.jp/Doya.htm` D. Kouznetvov: Doya function. (2011)

12) `http://tori.ils.uec.ac.jp/Tania.htm` D. Kouznetvov: Tania function. (2011)

13) Wenxue Li, Haifeng Pan, Liangen Ding, Heping Zeng, Wei Lu, Guangjun Zhao, Chengfeng Yan, Liangbi Su, and Jun Xu. Appl. Phys. Lett. **88** (2006), 221117.

14) D. Kouznetsov: Moscow University Physics Bulletin (2010), No.2, p.91

15) D. Kouznetsov, H. Trappmann: Mathematics of Computation **79** (2010), 1727.