Recovery of properties of a material from the transfer function of the bulk sample (theory)

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Abstract

Propertes (absorption, gain) of a medium are supposed to be functions of the intensity of light. For the straightforward characterization at a given intensity, either the physical model should be used (to extract from the measurements only the parameters of the model), or the sample should be thin, that allows to neglect variation of intensity within the example. The measurement of the transfer function of the bulk sample is suggested as an alternative. The transfer function of a sample determines the output intensity in terms of the input intensity. For the given transfer function, the distribution of intensity within the sample appears as the superfunction. Method for the recovery of the superfunction from the transfer function is suggested.

Keywords: Transfer equation, Transfer function, Superfunction, Abel function

1 Introduction

Investigations of the properties of the laser materials usually are related to some specific model. Consider the laser medium characterized with two main parameters, absorption A at the pump frequency and gain G at the lasing frequency; in quasi-stationary regime, these parameters are supposed to be functions of the intensity I_{pump} of pump and intensity I_{signal} at the laser frequency.

$$A = A(I_{\text{pump}}, I_{\text{signal}}) \ , \quad G = G(I_{\text{pump}}, I_{\text{signal}}) \tag{1}$$

Yet, there is no theory to get properties of these functions *ab inicio* in so explicit way, as it is done for the special mathematical functions [1]; the modeling and the experimental measurements are essential [2].

For the direct measurements of absorption and gain at given values of intensities, the sample should be optically thin. Then, the re-absorptoin can be neglected. However, each time, validity of such direct measurements should be investigated; the reabsorption may leads to mistakes [3, 4, 5, 6, 7].

This article suggests the tools that should significantly improve the precision of measurements of the parameters of the model (Section 2), as well as approximation of functions A and G, if the model of the medium is not yet established (Section 3). The tools allow to deal with optically-thick sample, that should significantly improve the precision of measurement. The transfer function of a uniformly-pumped sample should be measured and fitted. In the case of simple kinetics, the fitting function is known [8, 9, 10]; the algorithms for the evaluation are suggested and the numerical implementation is available. Even if the efficient model of excitations in some medium is not yet established, the can be extracted from the measurements of the transfer function of the opticallythick sample. The recovery is based on the formalism of superfunctions [13, 14] developed recently for the exponential transfer function [15, 16, 17] and generalized to other transfer functions [19, 18, 20].

Measurement of the transfer function for the optically-thick sample (and the recovery of the behavior of the intensity inside the sample through the formalism of superfunctions) should help to avoid mistakes with reabsorption and improve the precision of characterization of optical properties of the laser materials.

2 Example of a model of gain medium

A universal model valid for all laser types does not exist [2]. In the simplest case, the medium has two systems of sub-levels: upper and lower. They are characterized with effective cross-sections of absorption and emission at frequencies $\omega_{\rm p}$ and $\omega_{\rm s}$.

Let N be concentration of active centers in the solid-state lasers.
Let N_1 be concentration of active centers in the ground state.
Let N_2 be concentration of excited centers.
Let $N_1+N_2=N$, id est, no other states are taken into account.

The relative concentrations can be defined as $n_1 = N_1/N$ and $n_2 = N_2/N$.

The rate of transitions of an active center from ground state to the excited state can be expressed with

$$W_{\mathrm{u}} = \frac{I_{\mathrm{p}}\sigma_{\mathrm{ap}}}{\hbar\omega_{\mathrm{p}}} + \frac{I_{\mathrm{s}}\sigma_{\mathrm{as}}}{\hbar\omega_{\mathrm{s}}} \tag{2}$$

and the rate of transitions back to the ground state can be expressed with

$$W_{\mathrm{d}} = \frac{I_{\mathrm{p}}\sigma_{\mathrm{as}}}{\hbar\omega_{\mathrm{p}}} + \frac{I_{\mathrm{s}}\sigma_{\mathrm{es}}}{\hbar\omega_{\mathrm{s}}} + \frac{1}{\tau} \tag{3}$$

, where σ_{as} and σ_{ap} are effective cross-sections of absorption at the frequencies of the pump and the signal; σ_{es} and σ_{ep} are the same for stimulated emission; $\frac{1}{\tau}$ is rate of the spontaneous decay of the upper level.

Then, the kinetic equation for relative populations can be written as follows:

$$\frac{\mathrm{d}n_2}{\mathrm{d}t} = W_{\mathrm{u}}n_1 - W_{\mathrm{d}}n_2 \tag{4}$$

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} = -W_{\mathrm{u}}n_1 + W_{\mathrm{d}}n_2 \tag{5}$$

However, these equations keep $n_1 + n_2 = 1$.

The absorption A at the pump frequency and the gain G at the signal frequency can be expressed with

$$A = N_1\sigma_{\mathrm{pa}} - N_2\sigma_{\mathrm{pe}} \tag{6}$$

$$G = N_2\sigma_{\mathrm{se}} - N_1\sigma_{\mathrm{se}} \tag{7}$$

In many cases the gain medium works in a continuous-wave or in a quasi-continuous regime, causing the time derivatives of populations to be negligible. The steady-state solution can be written:

$$n_2 = \frac{W_{\mathrm{u}}}{W_{\mathrm{u}} + W_{\mathrm{d}}} \tag{8}$$

$$n_1 = \frac{W_{\mathrm{d}}}{W_{\mathrm{u}} + W_{\mathrm{d}}}. \tag{9}$$

The dynamic saturation intensities can be defined:

$$I_{\mathrm{po}} = \frac{\hbar\omega_{\mathrm{p}}}{(\sigma_{\mathrm{ap}} + \sigma_{\mathrm{ep}})\tau} \tag{10}$$

$$I_{\rm so} = \frac{\hbar\omega_{\rm s}}{(\sigma_{\rm as}+\sigma_{\rm es})\tau} \tag{11}$$

The absorption at strong signal:

$$A_0 = \frac{ND}{\sigma_{\rm as}+\sigma_{\rm es}} \tag{12}$$

The gain at strong pump:

$$G_0 = \frac{ND}{\sigma_{\rm ap}+\sigma_{\rm ep}} \tag{13}$$

where

$$D = \sigma_{\rm pa}\sigma_{\rm se} - \sigma_{\rm pe}\sigma_{\rm sa} \tag{14}$$

is determinant of cross-section.

At given intensities$I_{\rm p}$, $I_{\rm s}$ of pump and signal, the gain and absorption can be expressed as follows:

$$A = A_0 \frac{U+s}{1+p+s} \tag{15}$$

$$G = G_0 \frac{p-V}{1+p+s} \tag{16}$$

where

$$p = I_{\rm p}/I_{\rm po}\ , \quad s = I_{\rm s}/I_{\rm so} \tag{17}$$

$$U = \frac{(\sigma_{\rm as}+\sigma_{\rm es})\sigma_{\rm ap}}{D} \tag{18}$$

$$V = \frac{(\sigma_{\rm ap}+\sigma_{\rm ep})\sigma_{\rm as}}{D} \tag{19}$$

The following identities take place:

$$U - V = 1\ ,\ \ A/A_0 + G/G_0 = 1\ . \tag{20}$$

The state of gain medium can be characterized with a single parameter, such as population of the upper level, gain or absorption; gain never exceeds value G_0, and absorption never exceeds value A_0U.

Conversion of light in the gain medium can be characterized with the efficiency

$$E = \frac{I_{\rm s} G}{I_{\rm p} A} = \frac{\omega_{\rm s}}{\omega_{\rm p}} \frac{1 - V/p}{1 + U/s} \tag{21}$$

For the efficient operation, both intensities, pump and signal should exceed their saturation intensities: $\dfrac{p}{V} \gg 1$, and $\dfrac{s}{U} \gg 1$.

The consideration of propagation of signal at given pump can be described with variation of normalized intensity s on the longitudinal coordinate x, assuming that $s = s(x)$. Then

$$s'(x) = G(p, s(x))\, s(x) = G_0 \frac{p - v}{1 + p + s(x)} \tag{22}$$

While the normalized pump intensity p is supposed to be constant, it worth to define new "unsaturated" gain

$$G_1 = G_0 \frac{p - v}{1 + p} \tag{23}$$

and consider the new scale for the intensity of signal, let

$$S = s/(1 + p) \tag{24}$$

Then the equation for $S = S(x)$ can be written as follows:

$$S'(x) = G_1 \frac{S(x)}{1 + S(x)} \tag{25}$$

The solution can be written as

$$S(x) = \mathrm{Tania}(G_1(x - x_0)) \tag{26}$$

where, at least for real values of the argument,

$$\mathrm{Tania}(z) = \mathrm{LambertW}(\exp(z{+}1)) \tag{27}$$

Its inverse function can be expressed with

$$\mathrm{Tania}^{-1}(z) = \mathrm{ArcTania}(z) = z + \ln(z) - 1 \tag{28}$$

and x_0 is value of coordinate such that $S(x_0) = 1$. Properties of the Tania function are described at Citizendium [8] and TORI [9]; the efficient algorithm of the numerical evaluation is implemented. The transfer function T of an amplifier of length L can be described with

$$T(z) = \mathrm{Tania}\Big(G_1 L + \mathrm{ArcTania}(z)\Big) \tag{29}$$

and $S(x + L) = T(S(x))$.

Returning to the physical values, the transfer function T of an amplifier of length L can be described with

$$T(I) = J\ \mathrm{Tania}\Big(G_1 L + \mathrm{ArcTania}(I/J)\Big) = J\ \mathrm{Doya}^{G_1 L}(I/J) \tag{30}$$

where J is determined with pump (that is supposed to be constant),

$$J = I_{\mathrm{so}}/(1 + p) \tag{31}$$

and function Doya [10] can be expressed through the LambertW function as follows:

$$\mathrm{Doya}^q(z) = \mathrm{LambertW}\Big(z e^{z+q}\Big) \tag{32}$$

The efficient implementation of Doya function is available.

The table of measured T at various input intensities I can be fitted with function (30), giving the estimates for the parameters J and G_1.

In the similar way, the dependence of the transfer function for the pump intensity (at given signal intensity) can be analyzed, giving the accurate value of I_{po}. The job of experimental investigation of properties of the medium should be separated from the goal of achievement of the high efficiency of the laser action. The use of an external source of light at the lasing frequency may be a good decision for the analysis of dependence of A on the first argument at fixed values of the second argument.

Tracing the expressions (12),(13),(14) through the effective cross-sections , values σ_{s} and σ_{p} can be recovered. The fitting of the measured transfer function through the Tania function by (30) allows to work with the opticallythick samples, avoiding the wrong results due to neglecting of reabsorption at the conventional method of measurement of the effective cross-sections [3, 4, 5] discussed in [6, 7, 12].

The fitting of the transfer function of a bulk sample should be used for the precise measuring of the parameters of the model of the medium and for the careful test of the range of validity of the model. Up to my knowledge, even for the simple "two-level" media like the Ybdoped crystals, fibers and ceramics, such a test is not performed. Nobody knows, how many decimal digits are hold in the expressions (15) and (16) for the widely used optical materials. Hope, the fitting of the transfer functions will help to cover this gap in the research of properties of the laser materials.

3 Unknown model of the gain medium

For a new material, the model for the active medium may be not established. For example, in year 2008, it was not clear, wether the Mc-cumber relations apply for the "composite" materials such as $\mathrm{Yb} : \mathrm{Gd_2SiO_5}$ ceramics (some references are collected in [12]). In this case, the unknown dependence of the gain on the intensity of the amplified light also can be recovered with bulk materials, that allows the good precision. Such a recovery is based on the formalism of suoperfunctions, that first has been developed for the exponential transfer function [15, 16, 17] and then generalized to the transfer function of general kind [18, 19, 20]

Assume, one has to recover the gain as derivative of the intensity of light along the length of propagation, while it amplifies at the constant pump in the uniform medium. The naive approach is to use the optically thin sample and to measure the variation of the intensity from the input to the output. Evaluation of the small difference of two large quantities does not lead to the precise results. For this reason, it is suggested to measure the transfer function of the bulk sample. Such a measurement does not require to enter inside the sample to see what is happening during the amplification. The reconstruction of behavior of intensity inside can be done using the formalism of superfunctions [18, 19]. The algorithm for this recovery is suggested in this section.

Without loss of generality, we may use the thickness of sample as a unit of length. The intensity F of the signal inside the sample should satisfy the transfer equation

$$T(F(z)) = F(z+1) \tag{33}$$

Function T is supposed to be measured, id est, known. Function F should be recovered; the gain appears as derivative of the function. Solution F of the transfer equation (33) is called superfunction

From the first look, the problem is hopeless: the solution is not unique. If $F = f$ is the suerfunction, then another superfunction $F=\tilde f$ can be constructed with

$$\tilde f(z) = f(z+\eta(z)) \tag{34}$$

where η is holomorphic periodic function with period unity.

On the other hand, the modification from f to $\tilde f$ affects the analytical properties in the complex plane. Certain requirements on the behavior of the superfunction in the complex plane may drastically reduce the number of solution, as it happens with tetration [15, 16], giving hope to get namely physical solution.

Superfunction F is determined by its asymptotic behavior. It is supposed to approach the fixed points of the transfer function T. For the intensity of light in the gain medium (neglecting the spontaneous emission), the fixed point is zero; $T(0)=0$. Construction of a superfunction that approaches this value at minus infinity is described below.

Search the solution F of equation (33) in the following form:

$$F(z) = \varepsilon + a_2\,\varepsilon^2 + a_3\,\varepsilon^3 + ... \tag{35}$$

where $\varepsilon = \mathrm{e}^{kz}$; k and a_2, a_3,.. are constant coefficients; parameter e^k has sense of the coefficient of amplification a weak signal. Then

$$F(z+1) = k\varepsilon + a_2\,k^2\,\varepsilon^2 + a_3\,k^3\,\varepsilon^3 + ... \tag{36}$$

The substitution of (35) to the left hand side of the transfer equation (33) gives

$$T(F(z)) = T'\varepsilon + T'a_2\varepsilon^2 + T'a_3\varepsilon^3 + .. + \frac{T''}{2}(\varepsilon + a_2\varepsilon^2 + ..)^2 + \frac{T'''}{6}(\varepsilon + ..)^3 + .. \tag{37}$$

where $T'=T'(0)$, $T''=T''(0)$, $T'''=T'''(0)$. The substitution of (37) and (36) to (**??**) gives

$$\begin{aligned} \mathrm{e}^k &= T' && (38)\\ \mathrm{e}^{2k}a_2 &= T'a_2 + T''/2 && (39)\\ \mathrm{e}^{3k}a_3 &= T'a_3 + T''a_2 + T'''/6 && (40) \end{aligned}$$

and so on, determining values of k, a_2, a_3,.. . The representation (35) is good for the small values of ε. At positive k, the approxiamtion refers to the large negative z. For moderate values of z, the precise evaluation is possible with

$$F(z) = T^n\,(F(z-n)) \tag{41}$$

for some sufficiently large natural number n. Here, the upper superscript means the repeating iteration of operation, in the same way, as in quantum mechanics, the physical quantities are operators, and power of operator means its iteration, and does not refer to the exponentiation to base determined by the aplication of the operator to the wave function. In such a way, T^{-1} is inverse function of T, while $T^0(z)=z$, $T^1(z)=T(z)$, $T^2(z)=T(T(z))$, and so on.

The construction of superfunction F with (35)-(41) is called "regular iteration". The hypothesis is formulated [20] that for the realistic transfer function, regular iteration above gives namely the physically meaningful solution. The analysis of precise experimental data of measurement of the transfer function should allow to confirm (or to refute) this hypothesis.

While the hypothesis is verified [20] for the simple model transfer function

$$T^q(x) = \mathrm{LambertW}\big(x\ \mathrm{e}^{x+q}\big) \tag{42}$$

It is expected to work also for the experimental transfer functions. The confirmation (or negation) of this hypothesis should be greatly appreciated.

The knowledge of the transfer function T allows to construct the superfunction F. The inverse of the superfunction is the Abel function F^{-1}. Knowledge of the superfunction F and the Abel function F^{-1} allows to express the iterated transfer function; in the general case,

$$T^q(x) = F(q + F^{-1}(x)) \tag{43}$$

where number q of iterations has no need to be integer. From the transfer function of a sample of some fixed thickness, one can calculate the transfer function of arbitrary thickness.

The precise analysis of the behavior of samples in the working range of parameters could be used for analysis of the range of validity of the commonly used models. Also, the precise characterization would allow the prediction of phenomena of failure of the laser materials, for example, the avalanche upconversion [21], just from the data for the working range of parameters, without to expose the samples to the extremal pump power and without to destroy them. For such a characterization, the measurement of the transfer function and the recovery of superfunction should be useful tool.

Measurement of the transfer function of the optically-thick samples should greatly improve the precision of characterization of materials.

4 Conclusions

The tools for the investigation of the dependence of gain on the intensity are presented. These tools refer to the measurement of the transfer function of the optically thick sample.

For the case of known model of the gain medium, the only few parameters (saturation intensities, unsaturated gain) should be determined. In the simplest case, the measured transfer function should be fitted through the Doya function [10] with the expression (30).

For a new medium, the behavior of intensity inside the amplifier can be recovered from the transfer function with the *regular iteration* [16, 18], assuming the certain asymptotical behavior of superfunction determined with (35).

Measurement of the the transfer functions of the optically-thick samples and the treatment with formalism of superfunctions [13, 14] should improve the precision of characterization of a laser medium with the gain as function of intensity (1), avoiding the mistakes [4, 5, 6, 7] with reabsorption in the estimates of the cross-sections.

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