SELF-SPIKING AT LOW DAMPING

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Minimal model of the self-spiking of lasers.

Strong pump and low damping of self-oscillations

Solution of rate equations is expressed as quadrature.
(Unknown variable does not appear in the integrand.)

Width of peaks and the repetition rate
as functions of deepness of modulation.

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D. Kouznetsov, J.-F. Bisson, K. Takaichi, K. Ueda. High-power single mode solid state laser with short unstable cavity. – JOSA B, v.22, Issue 8, p.1605-1619 (2005).

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J.Dong, K.Ueda. Longitudinal-mode instabilities of Cr^{4+}, $Nd^{3+}:Y_3Al_5O_{12}$ self-Q-switched two-mode laser. Appl.Phys.Lett., 87, 151102 (2005).

X number of photons in the laser resonator
Y number of excitations active medium
$c_n \approx c/n_{\rm refr}$ is group velocity of light in the medium.
σ is the effective emission cross-section at the signal frequency.
L is transversal size of the mode of the laser mode
h is length of the resonator.
r is the transmission coefficient of the output coupler.
τ is lifetime or the excitation of the active medium.
P is power of pump absorbed in the active medium.

$$K = c_n\sigma/(L^2h) \qquad \frac{\mathrm{d}X}{\mathrm{d}t} = KXY - UX$$
$$U = c_n r/(2h)$$
$$V = 1/\tau \qquad \frac{\mathrm{d}Y}{\mathrm{d}t} = -KXY - VY + W$$
$$W = P/\hbar\omega_{\rm p}$$

Steady-state solution: $$X_{\rm o} = \frac{W}{U} - \frac{V}{K} \ , \quad Y_{\rm o} = \frac{U}{K}$$

The linearization of near the steady-state solution allows to estimate the decay rate ν and frequency Ω of self-pulsation:

$$\nu = KW/(2U) \ , \quad \Omega = \sqrt{KW - UV - \nu^2}$$

At low damping, Γ is small, and Ω is positive;

$$X = X_{\rm o} + X_{\rm o}\,\epsilon\,\cos(\Omega t)\ e^{-\Gamma t}$$

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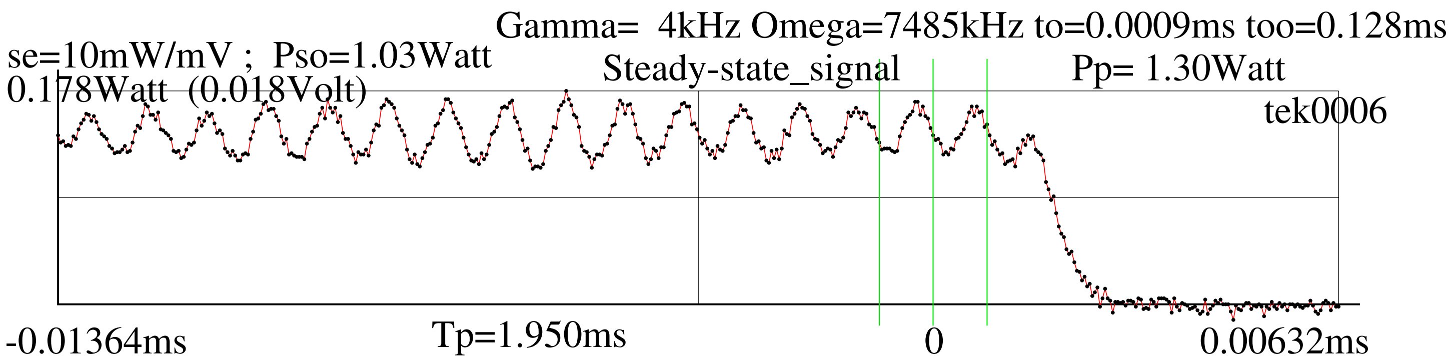
Example of the oscillogram:



As for the strong self-pulsation (spiking)..
A. Siegman. Lasers. – University Science Books, 1986.(p.958):
There do not seem to be, however, any simple analytic solutions to equations (25.1), (25.2) that apply during the period of strong spiking..

$$\frac{\mathrm{d}X}{\mathrm{d}t} = KXY - UX \quad , \qquad X_\mathrm{o} = W/U - V/K$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -KXY - VY + W \quad , \qquad Y_\mathrm{o} = U/K$$

Consider the new variables x, y; let

$$X = X_\mathrm{o} e^x$$

$$Y = Y_\mathrm{o} + X_\mathrm{o} y$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = K X_\mathrm{o} y$$

$$X_\mathrm{o} \frac{\mathrm{d}y}{\mathrm{d}t} = -e^x K X_\mathrm{o} Y_\mathrm{o} - K X_\mathrm{o}^2 Y - U Y_\mathrm{o} - V X_\mathrm{o} y + W$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \left(\frac{w^2}{U} e^x + V \right) \frac{\mathrm{d}x}{\mathrm{d}t} + w^2 \left(e^x - 1 \right) = 0 \quad , \qquad w = \sqrt{KW - UV}$$

$1/w$ determines the time-scale of the self-pulsation.

New variable $z = t\sqrt{w} = t\sqrt{WK - UV}$ (instead of time t)

$$\frac{\mathrm{d}^2 x}{\mathrm{d}z^2} + (\mu e^x + \nu)\frac{\mathrm{d}x}{\mathrm{d}z} - 1 + e^x = 0 \qquad \mu = w/U, \nu = V/w$$

For the case of strong spiking, $\mu \ll 1$, $\nu \ll 1$; then,

$$\frac{\mathrm{d}^2 x}{\mathrm{d}z^2} + -1 + e^x = 0$$

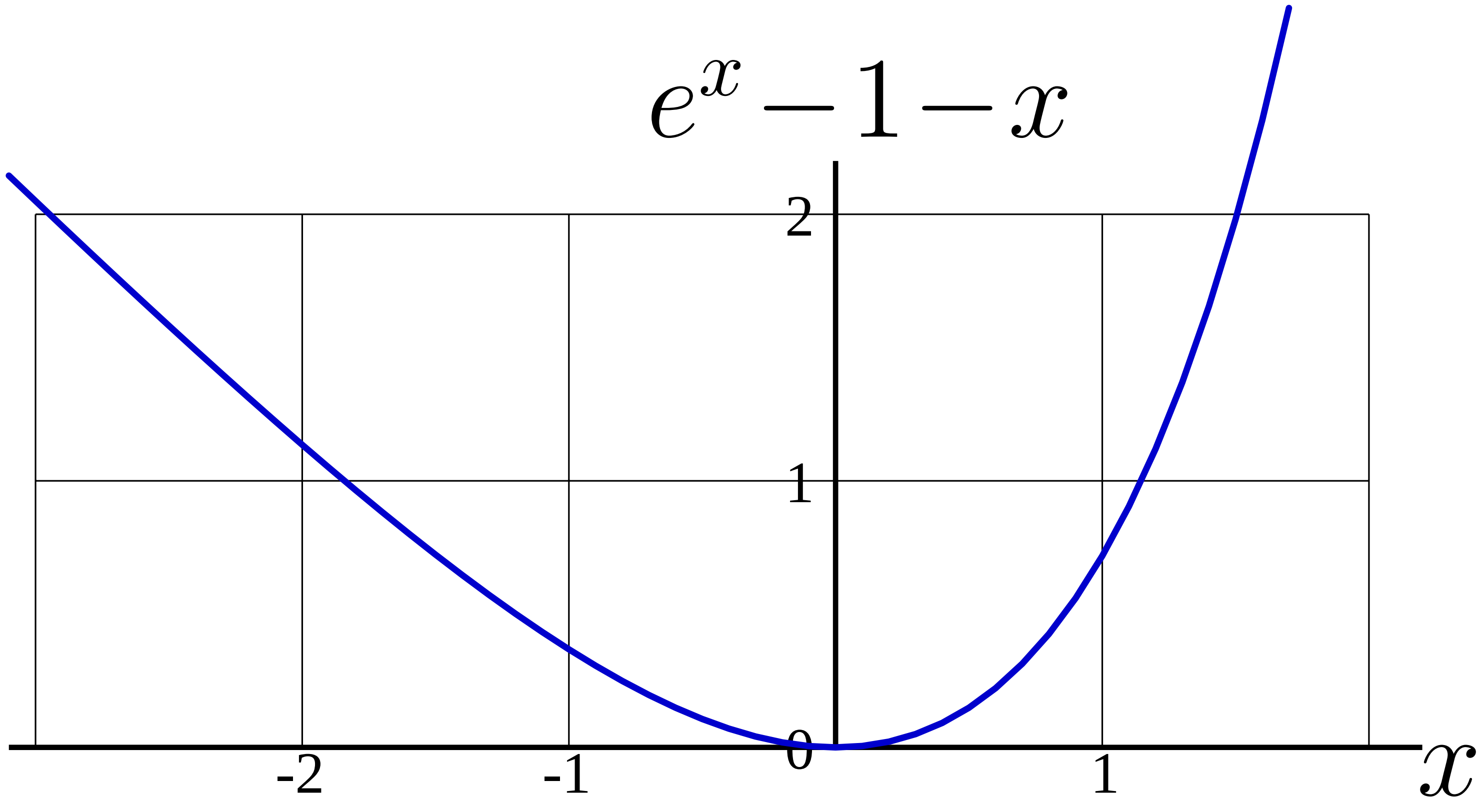
We have Classical nonlinear oscillator with unity mass;

$$\mathrm{Potential(x) = e^x - 1 - x}$$

Output power:

$$\mathrm{Power_{out}} = \frac{\hbar\omega_\mathrm{s}\theta}{2nch} X_\mathrm{o}\exp(x)$$

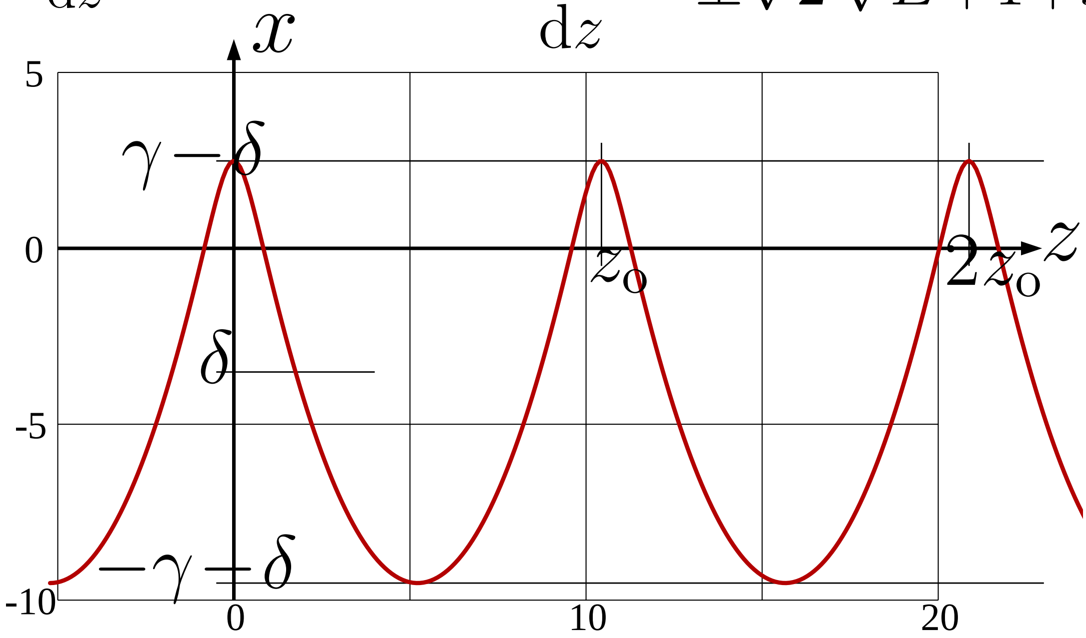
Classical oscillator with potential



Equation of evolution: $\frac{\mathrm{d}^2 x}{\mathrm{d}z^2} = 1 - e^x$

Conservation of quasi-energy: $\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + (e^x - x) = E = \text{constant}$

$$\frac{\mathrm{d}^2x}{\mathrm{d}z^2} = x - e^x \qquad \frac{\mathrm{d}x}{\mathrm{d}z} = \pm\sqrt{2}\sqrt{E+1+x-e^x}$$



Analytical solution:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\pm 1/\sqrt{2}}{\sqrt{E+1+x-e^x}} \quad ; \quad z = \frac{\pm 1}{\sqrt{2}} \int_x^{x_{\max}} \frac{\mathrm{d}a}{\sqrt{E+1+a-e^a}}$$

$$\gamma = \frac{x_{\max} - x_{\min}}{2} \quad , \qquad \delta = \frac{x_{\max} + x_{\min}}{2}$$

Then
$$X_{\min} = X_o \exp(x_{\min})$$
$$X_{\max} = X_o \exp(x_{\max})$$

Example: at $X_{\max}/X_{\min} = e^6 \approx 400$, $\gamma = 6$; the
The periodicity: it sufficient to analyze x at single period.

Maxima and minima of x

$$\begin{aligned} 2\gamma &= x_{\max} - x_{\min} \\ 2\delta &= x_{\max} + x_{\min} \end{aligned} \qquad \rightarrow \qquad \begin{aligned} x_{\max} &= -\delta + \gamma \\ x_{\min} &= -\delta - \gamma \end{aligned}$$

At maxima and at minima $\quad E = e^x - 1 - x$

$$e^{x_{\max}} - x_{\max} = e^{x_{\min}} - x_{\min}$$

$$x_{\max} - x_{\min} = e^{x_{\max}} - e^{x_{\min}}$$

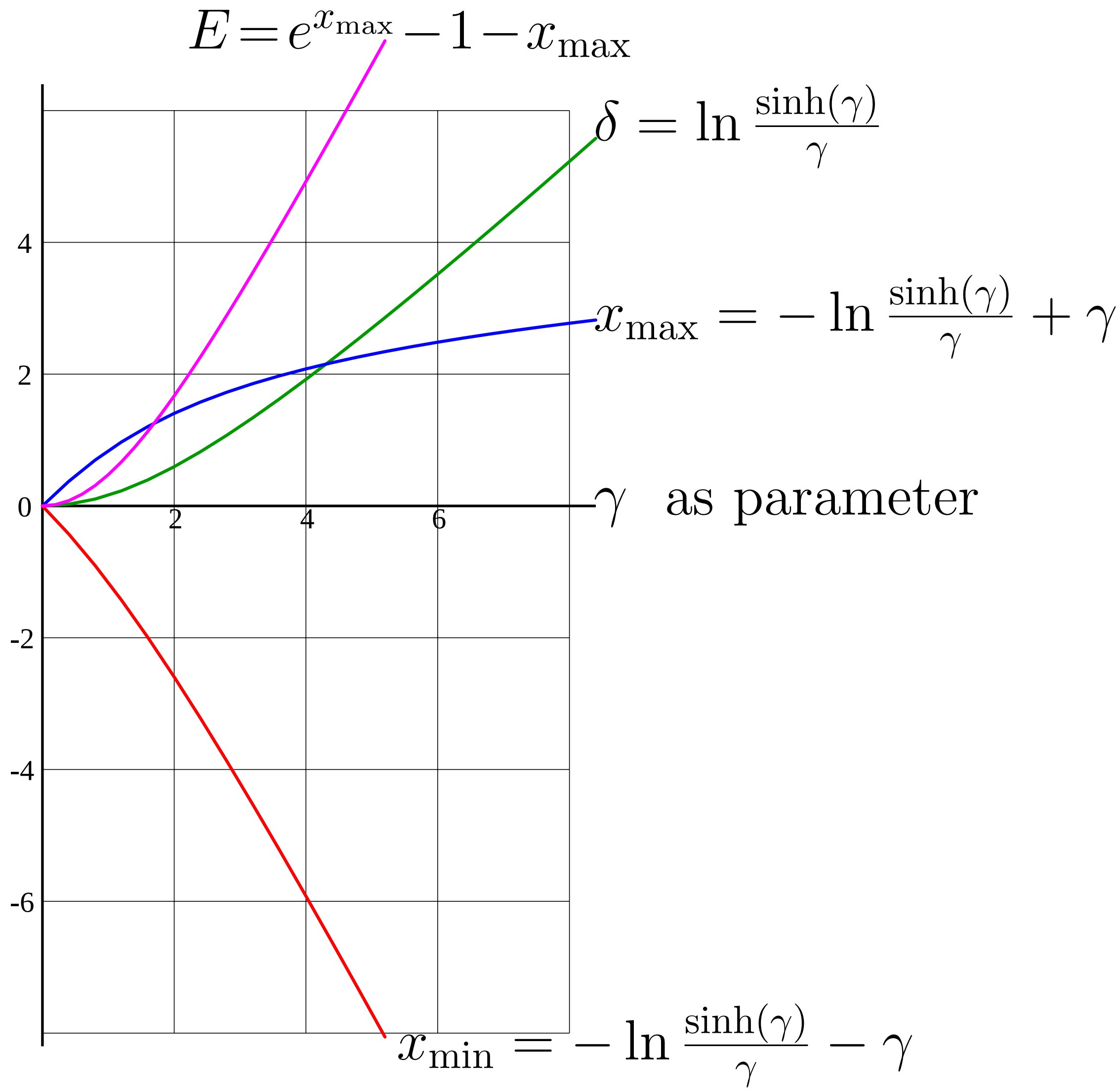
$$2\gamma = \exp(x_{\max}) - \exp(x_{\min}) = e^{-\delta}(e^{\gamma} - e^{-\gamma})$$

$$e^{\delta} = \sinh(\gamma)/\gamma$$

$$\delta = \ln \tfrac{\sinh(\gamma)}{\gamma}$$

$$x_{\max} = -\ln \tfrac{\sinh(\gamma)}{\gamma} + \gamma$$

$$x_{\min} = -\ln \tfrac{\sinh(\gamma)}{\gamma} - \gamma$$


$E = e^{x_{\max}} - 1 - x_{\max}$
$\delta = \ln \frac{\sinh(\gamma)}{\gamma}$
$x_{\max} = -\ln \frac{\sinh(\gamma)}{\gamma} + \gamma$
γ as parameter
$x_{\min} = -\ln \frac{\sinh(\gamma)}{\gamma} - \gamma$
4
2
0
-2
-4
-6
2
4
6

Period of pulsation

$$z_{\rm o} = 2\int_{x_{\rm min}}^{x_{\rm max}} \frac{{\rm d}a}{\sqrt{2}\ \sqrt{E+1+a-e^a}}$$

$$z_{\rm o} = \frac{1}{\sqrt{2}}\int_{x_{\rm min}}^{x_{\rm max}} \frac{{\rm d}a}{\sqrt{\exp(x_{\rm min})-x_{\rm min}+a-e^a}}$$

$$z_{\rm o} = \frac{1}{\sqrt{2}}\int_{-\delta-\gamma}^{-\delta+\gamma} \frac{{\rm d}a}{\sqrt{e^{-\delta-\gamma}+\delta+\gamma+a-e^a}}$$

$a = -\delta - \gamma\cos(\phi)$, ${\rm d}a = \gamma\sin(\phi){\rm d}\phi$

$$z_{\rm o} = z_{\rm o}(\gamma) = \frac{1}{\sqrt{2}}\int_0^{\pi} \frac{\gamma\ S\ {\rm d}\phi}{\sqrt{e^{-\delta-\gamma}-e^{-\delta-\gamma C}+\gamma-\gamma C}}$$

$C = \cos(\phi)$, $S = \sin(\phi)$

$$z_{\rm o}(\gamma) = \sqrt{2\frac{\sinh\gamma}{\gamma}}\int_0^{\pi} \frac{\gamma\ S\ {\rm d}\phi}{\sqrt{(1-C)\sinh\gamma - e^{-\gamma C}+e^{-\gamma}}}$$

period of pulsaiton $C = \cos(\phi) \quad , \quad S = \sin(\phi)$

$$z_{\mathrm{o}}(\gamma) = \sqrt{2\frac{\sinh\gamma}{\gamma}} \int_0^{\pi} \frac{\gamma \; S \; \mathrm{d}\phi}{\sqrt{(1-C)\sinh\gamma - e^{-\gamma C} + e^{-\gamma}}} \quad .$$

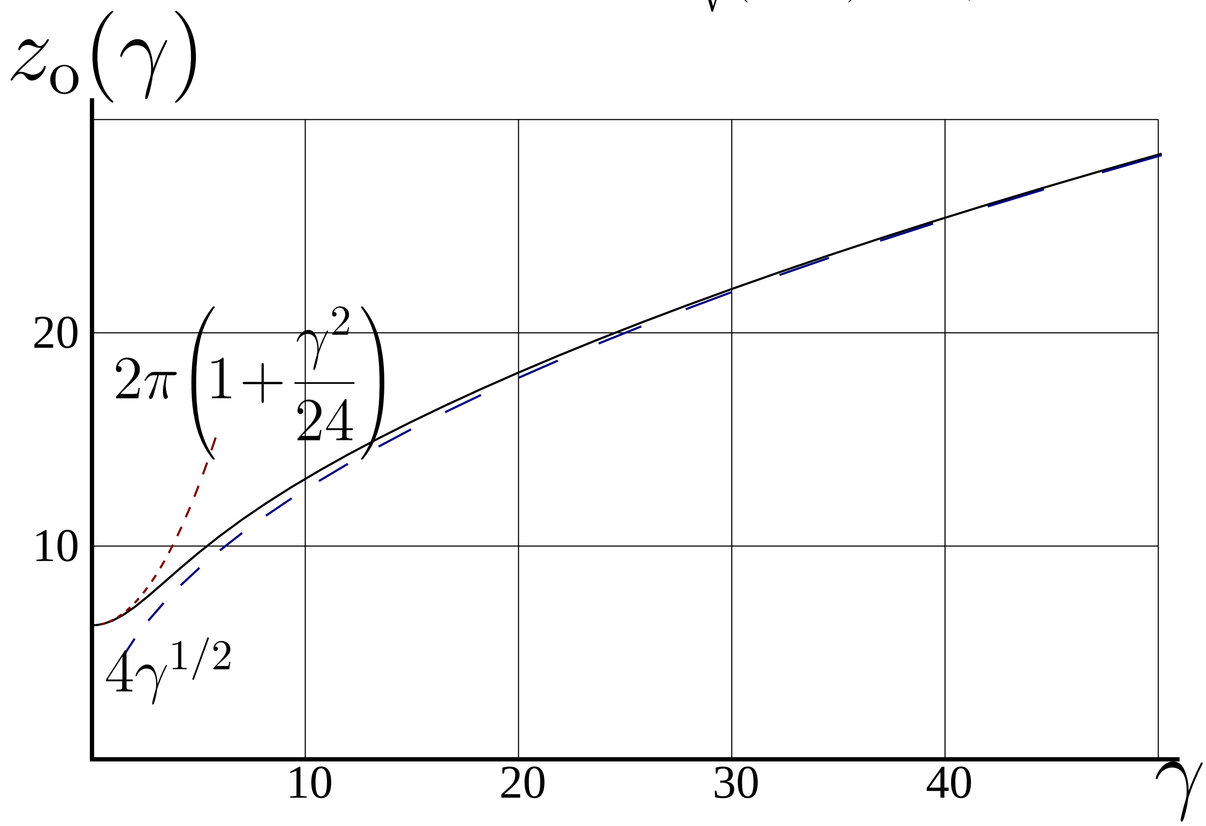
$$z_{\mathrm{o}}(\gamma) \approx 2\pi \left(1 + \tfrac{1}{24}\gamma^2 + \mathcal{O}(\gamma^4)\right)$$

$$z_{\mathrm{o}}(\gamma) \approx 4\sqrt{\gamma} + \mathcal{O}\left(\frac{\ln\gamma}{\sqrt{\gamma}}\right)$$

$$z_{\mathrm{fit}}(\gamma) = 2\pi\sqrt{\frac{\sqrt{400+107.38\gamma^2+8.685\gamma^4+0.05\gamma^5\ln(\gamma)+\frac{16}{\pi^4}\gamma^6}}{20+\gamma^2}}$$

$$\left|\frac{z_{\mathrm{fit}}-z_{\mathrm{o}}}{z_{\mathrm{o}}}\right| < .0004$$

$$z_o(\gamma) = \sqrt{2\frac{\sinh\gamma}{\gamma}} \int_0^\pi \frac{\gamma\ S\ \mathrm{d}\phi}{\sqrt{(1-C)\sinh\gamma - e^{-\gamma C} + e^{-\gamma}}}$$



Recover dimensions. Period $T = \frac{z_o(\gamma)}{\sqrt{w}} = \frac{z_o(\gamma)}{\sqrt{WK-UV}}$.

At $\gamma < 1$, $T \approx \frac{2\pi(1+\gamma^2/24)}{\sqrt{WK-UV}}$

Duration and shape of pulses

$$\frac{\mathrm{d}^2 x_{\mathrm{as}}}{\mathrm{d}z^2} = -e^{x_{\mathrm{as}}}$$

$$x_{\mathrm{as}} = \gamma - \delta - 2\ln(\cosh(bz))$$

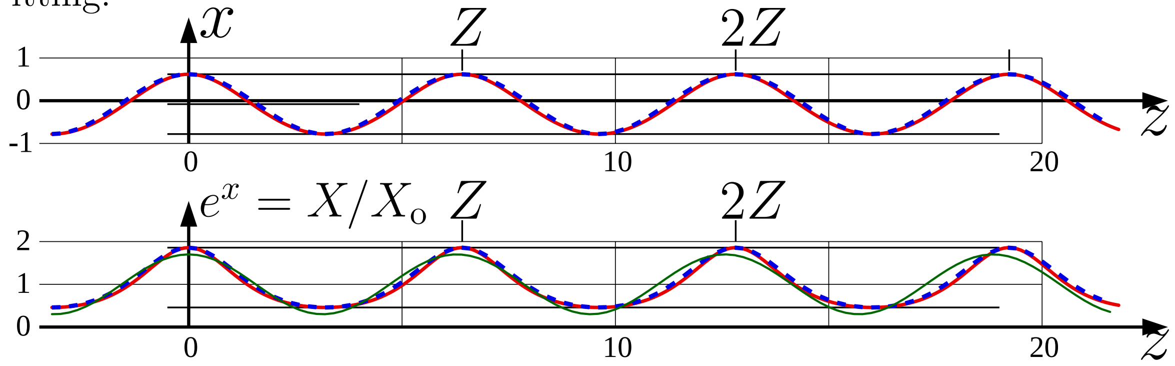
where

$$b = \sqrt{\tfrac{\gamma}{1-e^{-2\gamma}}} \approx \sqrt{\gamma}\left(1 + \mathcal{O}(e^{-2\gamma})\right)$$

Asymptotically, in vicinity of strong spikes,

$$X_{\mathrm{as}} = X_{\mathrm{o}} \tfrac{2b^2}{\cosh(bz)^2}$$

Fitting.

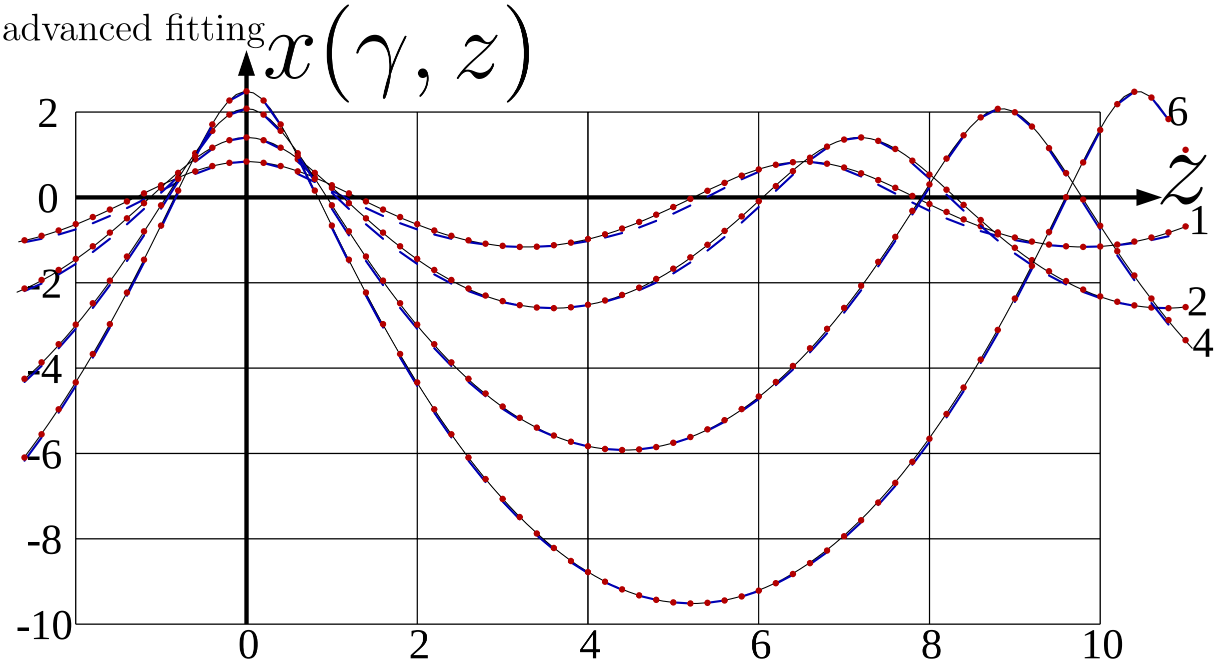


$x_\gamma(z)$ at $\gamma\!=\!0.7$, thick solid, and the cosinusoidal approximation, dashed.

$$x_0(\gamma, z) = -\delta + \gamma \cos(kz) \quad ,$$

$$k = k(\gamma) = 2\pi/Z(\gamma)$$

advanced fitting



x_5, dashed; x_6, dotted; $\gamma = 1, 2, 4, 6$.

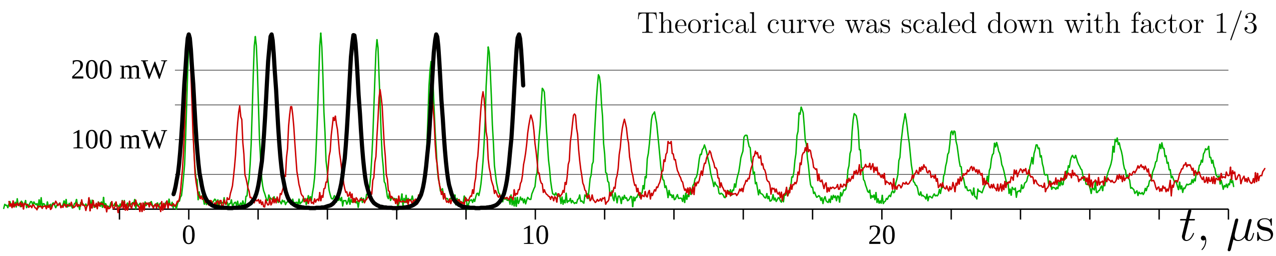
$$x_0(\gamma, z) = -\delta + \gamma\cos(kz) \quad , \qquad k = k(\gamma) = 2\pi/Z(\gamma)$$

$$x_5(\gamma, z) = \gamma - \delta - 2\ln\Big(\cosh\Big(\tfrac{b}{k}\sqrt{2(1-\cos(kz))}\Big)\Big) + (1-\cos(kz))\Big(\ln\Big(\cosh\tfrac{2b}{k}\Big) - \gamma\Big) \qquad \left(b = \sqrt{\frac{\gamma}{1-e^{-2\gamma}}}\right)$$

$$x_6(\gamma, z) = \frac{1}{1+\gamma^2}x_0(\gamma, x) + \frac{\gamma^2}{1+\gamma^2}x_5(\gamma, x) \ .$$

The deviation of x_6 does not exceed 1%.

I. COMPARISON WITH EXPERIMENTS



Yb:YAG microchip laser. 1-mm thick, 10% at. ytterbium-doped, $Y_3Al_5O_{12}$ ceramic disk obtained from Konoshima Chemical Corp.
A multilayer coating, antireflective at the pump wavelength $\lambda_{\rm p} = 940\,$nm, and highly reflective for the signal wavelengths $\lambda_{\rm s} = 1030\,$nm, is deposited on one face. The output coupler consists of a multilayer coating, transmission $\theta = 10\%$ transmission factor at the signal wavelength.
Laser operation of the device was performed by axial pumping through the rear mirror with a semiconductor laser diode (LIMO Corp.), delivering up to 26 W at $\lambda_{\rm p} = 940\,$nm. The pump was delivered with a 1-m long, 200-μm-diameter fiber with a numerical aperture NA = 0.22 . A pair of aspherical lenses with focal length $f = 8\,$mm and NA = 0.5 were used to concentrate the pump light on the microchip with a transmission efficiency of 96%.
The spot size was measured by the knife-edge method and found to be approximately Gaussian with a waist of about 80 μm.
The pump absorption efficiency was measured to be around 75%. The output power as function of time measured with the optical detector EG&G:G8194-32 and the Tektronix TDS 3052B oscilloscope.
Pulsed pumping was used to access the transient regime of this laser. Input pump power was reduced to 0.5 W in order to mitigate the thermal effects; this corresponds to absorbed power $P_{\rm p} = 375\,$mW. The pulse duration was 2 ms, with a 10-μs rising front.
assume $\sigma = 2\times 10^{-20}{\rm cm}^2$, $\tau = 0.97{\rm ms}$, $n = 1.8$.
Then $K = 5.3\times 10^{-6}\,$Hz, $U = 5.3\times 10^{9}\,$Hz, $V = 1030\,$Hz, $W = 1.77\times 10^{15}\,$Hz;
$X_{\rm o} = 1.4\times 10^8$, $Y_{\rm o} = 10^{15}$, $\Gamma = 1775\,$Hz ; $w = \sqrt{KW - UV} = 2\times 10^6\,$Hz $\approx \Omega$.
The maximum number of photons $X_{\max}$ is about 5 times larger than the average value $X_{\rm o}$, i.e. $X_{\max}/X_{\rm o} \approx 5$. Then $\gamma \approx 2.5$. period $T \approx 2.6\,\mu$s, which is similar to the period of pulsation in experimental data.
In the transient regime, the average output power is only about one third of what our model predicts.
We scale down the theoretical estimate with factor 1/3. This scaling means that our model misestimates the output power at the transient regime.

CONCLUSIONS Parameters: K, U, V, W

The number X of photons is proportional to the output power.

Self-pulsations are described with $\begin{cases} \mathrm{d}X/\mathrm{d}t = \ KXY - UX \\ \mathrm{d}Y/\mathrm{d}t = -KXY - VY + W \end{cases}$

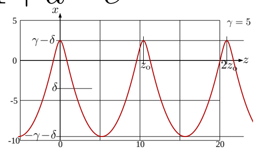
Steady-state solution: $X_{\rm o} = W/U - V/K \ , \quad Y_{\rm o} = U/K$

Low damping of self-pulsation, $u = U/(KX_{\rm o}) \gg 1 \ , \ U/V \gg 1.$

Weak pulsation with period $T = 2\pi/(KX_{\rm o})$ are known.

For strong spiking, only numerical solution is believed to exist.

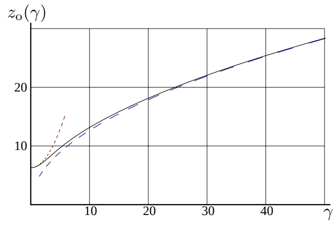
We have analytic solution: $z = \dfrac{\pm 1}{\sqrt{2}} \displaystyle\int_x^{x_{\max}} \frac{\mathrm{d}a}{\sqrt{E+1+a-e^a}}$



Logamplitude $\gamma = \ln(X_{\max}/X_{\min})$

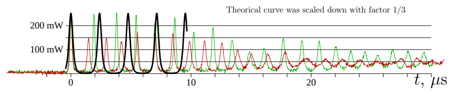
Then, the peak power $X_{\max} = X_0 e^{\gamma} \sinh(\gamma)/\gamma$;

and the drop power $X_{\max} = X_0 e^{-\gamma} \sinh(\gamma)/\gamma$



At the increase of logamplitude γ, the period of self-pulsation scales up with coefficient $z_{\rm o}(\gamma)/(2\pi)$.

The width of the pulses scales down as $\gamma^{-1/2}$.



Qualitative agreement with experiments.

Future work: The damping of oscillations. Distribution of gain.