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Quantum Noise in the Mapping of the Phase Space

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Abstract—Quadrature components of a single-mode field are interpreted as coordinates of the phase space. It is assumed that a quantum amplifier transforms the state of a single mode and the initial state of the field in this mode is a coherent squeezed one. The transfer function relating the average values of the field in the initial and final states determines mapping of the phase state. In the case of amplification, the field uncertainty in the final state is usually greater than in the initial state. This increase is interpreted as quantum noise of the amplifier. The lower bounds of this noise are estimated in terms of the derivatives of the transfer function. A degenerate parametric amplifier with pumping depletion is considered to be an illustration. Transformation of an initially orthogonal rectangular net in the phase space and deformation of the body of uncertainty given by the Wigner function are constructed for such an amplifier.

INTRODUCTION

A classical field in a certain mode can be amplified without introducing additional noise, whereas a quantum field cannot, even if prepared in a coherent state. Field amplification in a quantum mode produces quantum noise, i.e., increases the total uncertainty of quadrature field components.

We call an amplifier linear, if operators of the amplified field may be represented in the form of a linear combination of operators of the initial field with C numerical coefficients. The minimum noise of linear amplifiers is known [1–4]. But what is the minimum noise of nonlinear amplifiers? This question has as yet only been partially studied [5, 6].

For a phase-invariant amplifier, we have the lower bound of quantum noise [6] (phase invariance means that the amplification coefficient does not depend on the phase of the initial field). These estimates are applicable only for a field prepared initially in a coherent state. In this case, the lower bounds of noise of linear amplifiers [1–4] are applicable to any (not necessarily phase-invariant) linear amplifier and any initial state of the field. Needless to say, any partial estimate of the lower bound of quantum noise may be useful for properties of nonlinear amplifiers. But at the same time, it is desirable to have a stronger and more general estimate, where possible.

We showed that a phase-invariant nonlinear amplifier may produce a lower noise than an ideal linear amplifier with the same amplification coefficient [6]. In this paper, we generalize the lower bounds of noise in two ways. First, we consider an initial state with arbitrary squeezing. Second, we remove the requirement of phase invariance. Thus, our lower estimates are also applicable to parametric quantum amplifiers.

These two generalizations are related to each other. Noise in our mode may be determined as $D = D_1 + D_2 - 1/2$, where D_1 and D_2 are the dispersions of the quadrature components. When a parametric quantum amplifier is used, any of the dispersions D_1 or D_2 (but not both of them) may be as small as desired. In this case, the initial state of the field becomes squeezed. Thus, we should consider squeezing in order to obtain useful lower estimates for the noise of an amplifier of general type.

A parametric amplifier amplifies quadrature field components with different coefficients. Therefore, transformation of the average values of these components should be described by two functions or a single complex-valued transfer function. We will determine it in Section I. This function maps initial values of quadrature components onto their output mean values. The quadrature components of a single fixed mode correspond to the coordinates (x, p) of the phase space. Therefore, the transfer function determines mapping of the phase space onto itself. This is a nonlinear mapping in the general case. As already noted, our object is to bind possible values of quantum noise from below. In Section II, we derive the lower estimates for the noise D of an arbitrary amplifier and for the dispersions D_1 and D_2 of the quadrature components.

In Section III, we illustrate our results for the case of a parametric quantum amplifier with nonlinearity resulting from the depletion of pumping. We represent the transfer function as a distortion of the initially uniform rectangular net of values of initial quadrature components. To represent the growth of the uncertainty of the components in graphic form, we also construct the distribution of the Wigner function as the quasiprobability of the distribution of the quadrature components and demonstrate how the body of uncertainty is distorted when the phase space is subjected to nonlinear transformations.

I. AMPLIFICATION, COHERENT STATES, AND SQUEEZING

Quantum mechanics of amplifiers suggests that an amplifier converts an input field a into an output field A through a unitary transformation: $A = U^{+}aU$; we are using lower-case letters for the input field and upper-case letters for the output field. For simplicity, we restrict ourselves to the case of a single-mode amplifier. In terms of the operators of the input field a and its Hermite conjugation a^{+}, the input quadrature components may be represented as

$$a_1 \equiv \frac{1}{2}(a + a^{+}), \quad a_2 \equiv \frac{1}{2i}(a - a^{+}). \tag{1}$$

These components are coordinates of the phase space and do not commute. Let us denote the mathematical expectations of the input and output fields as $\langle a_i \rangle$ and $\langle A_i \rangle$, respectively. Then, U determines phase space mapping from $\langle a_1 \rangle$, $\langle a_2 \rangle$ to $\langle A_1 \rangle$, $\langle A_2 \rangle$. The transition from $\langle a_1 \rangle + i\langle a_2 \rangle$ to $\langle A_1 \rangle + i\langle A_2 \rangle$ determines the transfer function of the amplifier. The gain factor G_i is the ratio of the output and the input mathematical expectations, $G_i \equiv \langle A_i \rangle / \langle a_i \rangle$. In the general case, this factor is a function of both $\langle a_1 \rangle$ and $\langle a_2 \rangle$.

We determine the noise D of the amplified state as

$$D \equiv \langle A^{+}A \rangle - \langle A^{+} \rangle \langle A \rangle = D_1 + D_2 - \frac{1}{2}, \tag{2}$$

where

$$D_i \equiv \langle A_i^2 \rangle - \langle A_i \rangle^2 \quad (i = 1, 2). \tag{3}$$

The object of this work is to obtain the lower estimates for D_1, D_2, and D.

These estimates are known in certain particular cases. For a linear quantum amplifier, G_i is constant [1–4]. In this case, the uncertainties of the amplified quadrature components satisfy the inequality

$$D_1 D_2 \geq \frac{1}{16}(2G_1G_2 - 1)^2, \tag{4}$$

from which it follows that

$$D \geq G_1 G_2 - 1. \tag{5}$$

For $G_1 = G_2$, the amplifier is phase-invariant, and $D \geq G_2 - 1$. Both of these lower bounds (4) and (5) are true for any type of linear amplifier, regardless of the initial state of the field over which the mean values are taken. In the general (nonlinear) case, lower estimates require some knowledge of the initial state, and the minimum noise is no longer determined by the value of the gain factor for a given initial state. In the previous work [6], lower estimates of the noise of a phase-invariant amplifier are presented under the assumption that the field is initially prepared in a coherent state. Here, we allow for squeezing of the initial state $|\rangle$

$$|\rangle = |\alpha, 0\rangle_z \equiv T_\alpha S_z |0, 0\rangle, \tag{6}$$

where T_α is the translation operator

$$T_\alpha \equiv \exp(\alpha a^{+} - \alpha^* a), \tag{7}$$

and S_z is the squeezing operator

$$S_z \equiv \exp[z(a^{+})^2/2 - z^* a^2/2]. \tag{8}$$

Here, α and z are C numerical complex-valued parameters, α determines the amplitude and the phase of the initial coherent state, and z determines the direction and degree of squeezing; we are using Greek letters to denote coherent states, and Latin letters to denote n-photon states. The state $|m, 0\rangle$ represents m photons and a certain specially prepared state of the amplifier uncorrelated to them. Although this notation could suggest that the amplifier is prepared in the ground state, we do not make such an assumption. In what follows, an expression $\langle Q \rangle$ for any operator Q determines the mathematical expectation of the operator Q over the state $|\alpha, 0\rangle_z$.

One should rewrite the state $|\alpha, 0\rangle_z$ in a somewhat different form. Using the identity

$$S_z^{+} a S_z = a\cosh|z| + a^{+}\frac{z}{|z|}\sinh|z|, \tag{9}$$

we can change the order of the translation and the squeezing operators in formula (6). We have

$$|\alpha, 0\rangle_z = T_\alpha S_z |0, 0\rangle = S_z T_\beta |0, 0\rangle, \tag{10}$$

where

$$\beta = \alpha\cosh|z| - \alpha^*\frac{z}{|z|}\sinh|z|. \tag{11}$$

II. LOWER BOUNDS OF NOISE AND DISPERSIONS

For the lower bounds of the noise of nonlinear amplifiers to be obtained, two basic formulas are required. First, it follows from formulas (1) and (17) of [6] that

$$\frac{1}{\sqrt{m!}}\frac{\partial^m}{\partial \beta^m}\langle \beta, 0|Q|\beta, 0\rangle = \langle 0, 0|T_\beta^{+} Q T_\beta|m, 0\rangle \tag{12}$$

for any Q independent of β (here, $|\beta, 0\rangle$ represents a coherent state of the field).

Second, note that the operator

$$I_0 = \sum_{m=0}^{\infty} S_z T_\beta |m, 0\rangle\langle m, 0| T_\beta^{+} S_z^{+} \tag{13}$$

is a projector $I_0^2 = I_0$ and inequality $\langle \text{any}|\text{any}\rangle \geq \langle \text{any}|I_0 \text{any}\rangle$ is valid for any state. In particular,

$$\langle A^{+}A \rangle \geq \langle A^{+} I_0 A \rangle. \tag{14}$$

From formulas (12)–(14), follows

Theorem 1. $D \geq E(\alpha, z)$ and $D \geq F(\alpha, z)$, where

$$(\alpha, z) = \sum_{n=1}^{\infty} \frac{1}{n!}\left|\left(\cosh|z|\frac{\partial}{\partial\alpha} + \frac{z^*}{|z|}\sinh|z|\frac{\partial}{\partial\alpha^*}\right)^n \langle A^+\rangle\right. \tag{15}$$

and

$$F(\alpha, z)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n!}\left|\left(\cosh|z|\frac{\partial}{\partial\alpha} + \frac{z^*}{|z|}\sinh|z|\frac{\partial}{\partial\alpha^*}\right)^n \langle A\rangle\right|^2 - 1. \tag{16}$$

The Proof. Let us substitute formulas (14) in definition (2) of the noise D to obtain

$$D \geq \langle A^+ I_0 A\rangle - \langle A^+\rangle\langle A\rangle$$
$$= \sum_{m=1}^{\infty} |\langle 0, 0|T_\beta^+ S_z^+ A^+ S_z T_\beta|m, 0\rangle|^2. \tag{17}$$

Let us use (12) for $Q = S_z^+ A^+ S_z$. This gives

$$D \geq \sum_{m=1}^{\infty} \frac{1}{m!}\left|\frac{\partial^m}{\partial\beta^m}\langle 0, 0|T_\beta^+ S_z^+ A^+ S_z T_\beta|0, 0\rangle\right|^2$$
$$= \sum_{m=1}^{\infty} \frac{1}{m!}\left|\left(\cosh|z|\frac{\partial}{\partial\alpha} + \frac{z^*}{|z|}\sinh|z|\frac{\partial}{\partial\alpha^*}\right)^m \langle A^+\rangle\right|^2 \tag{18}$$
$$= E(\alpha, z).$$

However, we may utilize the conjugation of formula (14). In this case, A and A^+ change places. Let us apply the relation $AA^+ = A^+A - 1$ to definition (2). Then, using (12) for $Q = S_z^+ A S_z$, we get the lower estimate for $F(\alpha, z)$.

Let us apply this theorem to special cases. By setting the squeezing parameter $z = 0$ and assuming that $G = G_1 = G_2$ depends only on $\alpha^*\alpha$, we obtain conditions for a nonlinear phase-invariant amplifier. In this case, the output mean value of the field is $\langle A\rangle = G\alpha$, and Theorem 1 reproduces the lower estimate of the noise of such an amplifier [6].

In a still more special case, we set G = const and obtain the lower bound [1–4] for a linear phase-invariant amplifier.

Let us turn now to the dispersions D_1 and D_2 of the quadrature components. For a linear parametric amplifier, the noise in one component may be as small as desired. For a nonlinear parametric amplifier, we have

Theorem 2. $D_i \geq E_i(\alpha, z)$, where

$$E_i = \sum_{n=1}^{\infty} \frac{1}{n!}\left|\left(\frac{1}{2}\left[\cosh|z| + \frac{z^*}{|z|}\sinh|z|\right]\frac{\partial}{\partial\alpha_1}\right.\right.$$
$$\left.\left.+ \frac{1}{2i}\left[\cosh|z| - \frac{z^*}{|z|}\sinh|z|\right]\frac{\partial}{\partial\alpha_2}\right)^n \langle A_i\rangle\right|^2. \tag{19}$$

The Proof. Similar to the proof of Theorem 1, we start with the definition $D_i = \langle A_i^2\rangle - \langle A_i\rangle^2$ and have

$$D_i \geq \langle A_i I_0 A_i\rangle - \langle A_i\rangle^2$$
$$= \sum_{m=1}^{\infty} |\langle 0, 0|T_\beta^+ S_z^+ A_i S_z T_\beta|m, 0\rangle|^2. \tag{20}$$

Taking (12) into account, we obtain

$$D_i \geq \sum_{m=1}^{\infty} \frac{1}{m!}\left|\left(\cosh|z|\frac{\partial}{\partial\alpha} + \frac{z}{|z|}\sinh|z|\frac{\partial}{\partial\alpha^*}\right)^m \langle A_i\rangle\right|^2$$
$$= \sum_{m=1}^{\infty} \frac{1}{m!}\left|\left(\frac{1}{2}\left[\cosh|z| + \frac{z^*}{|z|}\sinh|z|\right]\frac{\partial}{\partial\alpha_1}\right.\right. \tag{21}$$
$$\left.\left.+ \frac{1}{2i}\left[\cosh|z| - \frac{z^*}{|z|}\sinh|z|\right]\frac{\partial}{\partial\alpha_2}\right)^m \langle A_i\rangle\right|^2 = E_i(\alpha, z).$$

Note that only the first term ($m = 1$) in expression (19) for $E_i(\alpha, z)$ survives for a linear parametric amplifier. In a proper choice of the squeezing parameter z, any of the dispersions D_1 or D_2 may be made as small as desired. By choosing z, which makes D_1 small, we make D_2 large at the same time, and *vice versa*. Therefore, we cannot make both D_1 and D_2 as small as desired while retaining equation (4). To extend this equation to the case of nonlinear parametric amplifiers, we could multiply the estimates $D_1 \geq E_1(\alpha, z)$ and $D_2 \geq E_2(\alpha, z)$ of Theorem 2 and obtain $D_1 D_2 \geq E_1(\alpha, z)E_2(\alpha, z)$, but this would not be the best possible estimate. It does not reproduce formula (4) in the linear case [7].

III. EXAMPLE

All real amplifiers are saturable. Saturation manifests itself in the fact that the gain factors G_i become functions of input amplitudes α_1 and α_2, and the mapping of the phase space becomes nonlinear. As an example of such nonlinear mapping, we consider a parametric amplifier with depleted pumping [8]

$$H = \frac{1}{2i}[a^2 b^+ - (a^+)^2 b]; \tag{22}$$

$U = \exp(-iHt)$ determines unitary transformation. Here, b is the operator of field in the pumping mode with the same commutation relations as in mode a; a and a^+ commute with b and b^+. Assume that the initial state is coherent in each mode: $\alpha = \langle a\rangle$, $\beta = \langle b\rangle$. The transfer function maps α_1 and α_2 into $\langle A_1\rangle$ and $\langle A_2\rangle$.

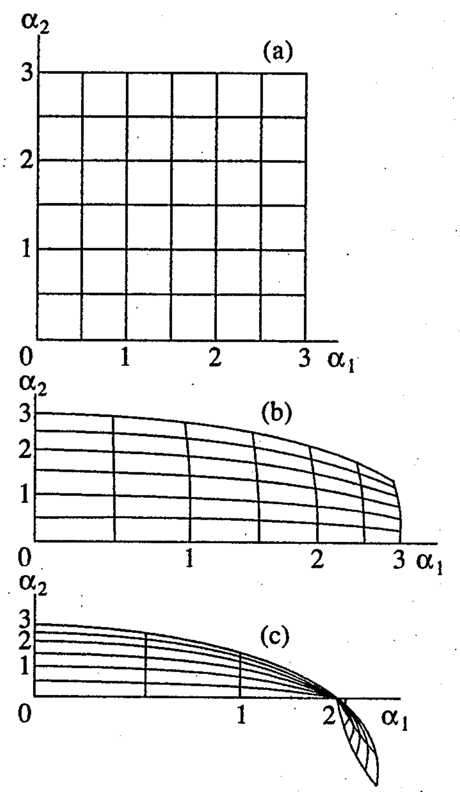

Fig. 1. Transformation of the initial rectangular net α_1 = const and α_2 = const in a parametric amplifier for t = (a) 0, (b) 0.3, and (c) 0.5.

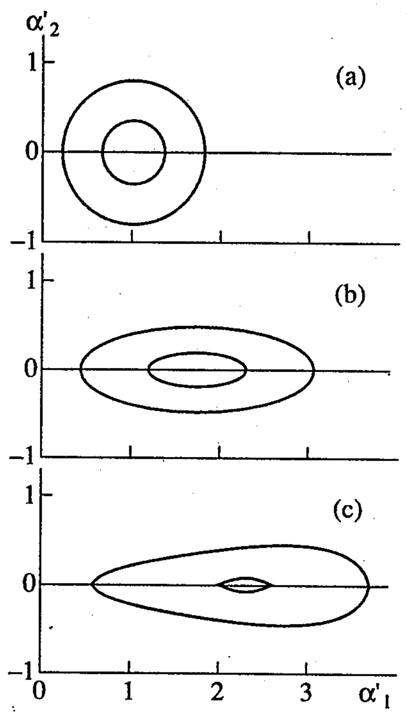

Fig. 2. Deformation of the Wigner function of the initial coherent state with $\alpha = 1$ for t = (a) 0, (b) 0.3, and (c) 0.5. The level lines $W = 0.3$ and $W = 0.8$ are shown.

For numerical calculations illustrating mapping of the phase space, we diagonalized the Hamiltonian on the subspace of states with a number of photons no greater than 40. In doing so, we considered states $|k-2m, m\rangle$ with $0 \leq k \leq 40$ and $0 \leq m \leq k/2$ (k and m are integers) in the Fock representation. Here, $a^+a|k-2m, m\rangle = (k-2m)|k-2m, m\rangle$, and $b^+b|k-2m, m\rangle = m|k-2m, m\rangle$. Note that transformation U does not mix states with different values of k, so that, in fact, we did not need to diagonalize matrices wider than 21×21.

The MAPLE computer code was used for diagonalization, and eigenvalues and eigenfunctions were calculated with 20 significant digits.

We chose $\beta = 2$ to show how nonlinear parametric amplification deforms the initial phase space. Let us have a uniform orthogonal net for $t = 0$ (Fig. 1a). The quality of representation allows a deformation of this net to be seen in the right upper corner. It is associated with finiteness of the subspace on which the Hamiltonian is diagonalized. Figures 1b and 1c show the deformation of this net for $t = 0.3$ and 0.5. These mappings are symmetric with respect to $\alpha_1 \longrightarrow -\alpha_1$ and $\alpha_2 \longrightarrow -\alpha_2$, so only the first quadrant of the coordinate plane is presented in the figures. For $t = 0.3$, the mapping is almost linear for $|\alpha| < 2$. Coordinate squares are converted into prolate rectangles typical of linear squeezing. For larger values of the initial field, these rectangles are deformed. For $t = 0.5$, this mapping becomes ambiguous.

The uncertainty of the output field can be characterized in more detail by the Wigner function

$$W(\alpha_1', \alpha_2') = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_{n,m}^* c_{k,m} \times \int_{-\infty}^{\infty} \phi_n\left(\alpha_1'\sqrt{2} - \frac{u}{2}\right)\phi_k\left(\alpha_1'\sqrt{2} + \frac{u}{2}\right)e^{i\sqrt{2}\alpha_2' u}du, \quad (23)$$

where $c_{n,m}$ are coefficients of expansion of a transformed state in states with n photons of field and m photons of pumping, $\phi_n(x) = H_n(x)e^{-x^2/2}/\sqrt{2^n n!\sqrt{\pi}}$, and $H_n(x)$ are Hermitian polynomials [9].

The initial state for $\alpha = 1$ is shown in Fig. 2a by concentric circles centered at the point $\alpha_1' = \alpha_1$, $\alpha_2' = \alpha_2$ (we construct the Wigner function in the coordinates α_1', α_2', which correspond to $x/\sqrt{2}$, $p/\sqrt{2}$). For $t = 0.3$, these circles are deformed. They look like ellipses, and this deformation corresponds to linear compression of the net in Fig. 1b.

For large t, the right part of the body of uncertainty becomes wider than the left one (Fig. 2c).

Note than $A_1 = G_1 a_1$ and $A_2 = G_2 a_2$, and for a linear parametric amplifier, the Wigner function W of an amplified state may be expressed in terms of the Wigner function w of the initial state: $W(\alpha_1', \alpha_2') = w(\langle A_1\rangle, \langle A_2\rangle)$ for $\langle A_1\rangle$ and $\langle A_2\rangle$, which is determined by the transfer

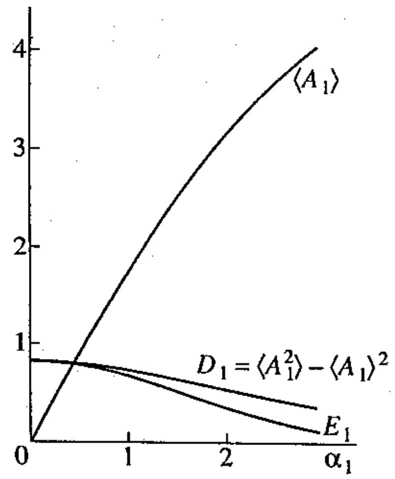

Fig. 3. The real part of the transfer function $\langle A_1 \rangle$ for $\alpha_2 = 0$ and $t = 0.3$ and the dispersion D_1 and its lower bound E_1 for this transfer function.

function for the input values α_1' and α_2'. One needs to compare Figs. 1c and 2c to realize this. The net in Fig. 1c is condensed to the right, whereas the body of uncertainty becomes wider. Only the mean position of the body of uncertainty follows the mapping of the transfer function of the initial position. The nonlinear behavior of the body of uncertainty demonstrates how the second and higher derivatives of the transfer function increase the noise of an amplifier.

Let us consider an amplifier with the transfer function shown in Fig. 2b. Figure 3 presents $\langle A_1 \rangle$ as a function of α_1 for $\alpha_2 = 0$. Only the first quadrature component of the field is amplified in such an amplifier. Let us examine the dispersion of this component. This dispersion is plotted in the same diagram in comparison with the lower estimate in accordance with Theorem 2. The gain factor G_1, the dispersion D_1, and its lower bound E_1 decrease as the amplitude grows, but D_1 remains greater than its lower bound, as must happen.

Compression of the profile of uncertainty in the region of net compression can be interpreted semiclassically. The mapping corresponding to degenerate parametric amplification does not change an element of the phase volume $dpdx$, so it can be carried out without additional degrees of freedom [equation (23) with a C-number instead of b describes a parametric amplifier with classical pumping]. But compressions or dilations of the coordinate and momentum in which the phase volume is not conserved (as on the right side of Fig. 1b) suggest the correlation of field states with additional degrees of freedom. Such a correlation appears as additional quantum noise.

CONCLUSIONS

We characterized a single-mode amplifier of general type by its transfer function, which is defined in the phase space. This function maps mathematical expectations of the quadrature components of the initial field onto mathematical expectations of the components of the output field. Saturation of amplification makes this function nonlinear. Such mapping generates quantum noise, and we derived the lower bound of this noise for the total noise and the dispersion of each quadrature component in Theorems 1 and 2. Theorem 1 gives the lower bound of the noise of a depleted parametric amplifier with a squeezed coherent input signal. For such an amplifier, Theorem 2 gives the lower bound of the dispersion of a single quadrature component of the amplified field. These are the first lower bounds of quantum noise of a nonlinear amplifier of general type.

Particular examples show that the lower bounds given by Theorems 1 and 2 are valid, although the body of uncertainty and phase space are not always deformed in the same manner.

For simplicity, we considered states of a single mode of the field. Multimode states may also be important in practice. The lower bounds given by Theorems 1 and 2 are not the best possible, because the projector I_0 introduced in (13) is only one of a set of orthogonal projectors in the multimode space. The consideration of additional projectors may improve the lower bounds.

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