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Holomorphic asymptotics

Asymptotic expansions of holomorphic functions are considered. Examples are suggested and supplied with tables, explicit plots and complex maps.

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Status.

This manuscript is preliminary. Definitions, terminology, and examples are subject to revision.

Attribution

ChatGPT (OpenAI) contributed to the editorial structuring, terminology alignment, and refactoring of several chapters, including the formulation of asymptotic definitions and consistency checks of mathematical exposition.

License

Preface

This book is devoted to the theory and computation of special functions, with emphasis on asymptotic methods and numerical evaluation. The material grew out of the TORI project.

Contents

Chapter 1

Prologue

This book is motivated by the attempts to load at server mizugadro the series of articles with definitions of terms related to the asymptotical analysis, «Asymptotic», «Sectorial asymptotic», «Strip Asymptotic». These concepts are imlicitely used in Mathematics and Physics at least since century 20, but the definitions found at the free access sites seem to be not sufficient. Among the sources available in the free access, the most complete seem to be the descriptions by Frank W.J. Olver and R. Wong [1, 2].

In book «Superfunctions» [3, 4], the concepts mentioned are used well beyond of the formal definitions. The euristic style of that book implies, that some definitions and deductions are omitted - over-vice, the book would be of size of "Abramotits-Stegun",

This book appears ad bunch of definitions; all other can be considered as illustrations. I try to follow the terminology from the publications cited, although the definitions are designed to be used for complex values of the arguments of the functions.

Some colleagues qualify many things as «obvious» - until some contradictions appear.

In the informal discussion, at some scientific and near-scientific slang, words «by definition» are used without referring to any definition of the term discussed. Practically, in such a slang, words «by definition» mean «I do not know and I do not want to know he definition of this term, but I want you to accept my statement without discussion.» Apparently, such an approach refers to the «motivated reasoning», so-called «Female logic» [5, 6].

I do not criticize such an approach. Perhaps, sometimes it is useful and even unavoidable in an euristical approach. But I still try to define the terms.

Chapter 2

Introduction

The goal of the scientific treatment of a function, and in particular of a holomorphic function, is its conversion into a special function.
In this book, the term “special function” is used in an operational sense.

1 Special Functions

A function f is qualified as a special function if the following conditions are satisfied:

1. The definition of f is available in free access.
2. The properties of f are investigated and documented in free access; in particular, its asymptotic behavior is described.
3. Relations of f with other special functions are known.
4. An efficient algorithm for the evaluation of f is implemented and available in free access.

The efficient evaluation of special functions is often achieved through asymptotic approximations.

2 Evaluation and Approximation

In this book, a clear distinction is made between evaluation and approximation.
Evaluation is understood as an action or procedure that produces a numerical value of a quantity. The result of an evaluation may be exact or approximate.
An approximation is a function or expression intended to be close to another function within a specified domain.

An asymptotic is a special type of approximation whose quality improves in a limiting regime and whose validity depends on the domain in which the limit is taken.

3 Role of the Domain

The asymptotic behavior of a function cannot be characterized independently of the domain of its input variable.
Different domains may lead to different asymptotic interpretations of the same function.
This is particularly important for functions of a complex variable, where sectorial and restricted asymptotics naturally arise.

4 Motivation

In many practical problems, the terms "asymptotic", "approximation", "estimate", and "evaluation" are used informally and sometimes interchangeably.
Such usage may be sufficient in heuristic discussions, but it becomes problematic in the context of numerical implementation and rigorous analysis.
The purpose of this book is to introduce a consistent terminology and to illustrate it by explicit examples, with particular emphasis on holomorphic functions and their asymptotic behavior.

Chapter 3

Asymptotic Definitions

This chapter introduces definitions related to asymptotic behavior. The emphasis is on explicit specification of the domain in which the asymptotic relations are considered.

1 Asymptotic Approximation

Let f and A be functions defined on a set D, and let z_0 be a limit point of D.
The expression

$$f(z) \sim A(z) \quad \text{as } z \to z_0 \text{ in } D$$

means that

$$\lim_{\substack{z \to z_0 \\ z \in D}} \frac{f(z)}{A(z)} = 1.$$

The specification of the set D is an essential part of the asymptotic statement.

2 Restricted Asymptotics

An asymptotic relation is called restricted if the limit $z \to z_0$ is taken only within a proper subset D of the natural domain of the functions involved.
Different choices of D may lead to different asymptotic interpretations of the same functions.

3 Sectorial Asymptotics

Let $z_0 = \infty$. A sectorial asymptotic is an asymptotic relation in which the set D is a sector

$$D = \{z \in \mathbb{C} : |z| > R,\ \alpha < \arg z < \beta\}.$$

Sectorial asymptotics are typical for holomorphic functions with branch points or Stokes lines.

4 Asymptotic Expansion

An asymptotic expansion of a function f is a formal series

$$A(z) = \sum_{k=0}^{\infty} a_k \phi_k(z)$$

such that for each n,

$$f(z) - \sum_{k=0}^{n-1} a_k \phi_k(z) = o\big(\phi_n(z)\big) \quad \text{as } z \to z_0 \text{ in } D.$$

The functions ϕ_k are ordered by decreasing magnitude in the limit considered.

5 Uniformity

An asymptotic relation is called uniform with respect to a parameter if the convergence in the asymptotic limit is uniform in that parameter. Non-uniform asymptotics often require restriction of the domain or the use of different asymptotic representations in different regions.

6 Remarks

- An asymptotic approximation is not required to converge.
- An asymptotic expansion may provide highly accurate numerical evaluations when truncated appropriately.
- The same function may admit different asymptotic representations in different domains.

Chapter 4

An Elementary Example with Domain Restriction

This chapter illustrates that the notion of an asymptotic cannot be separated from the specification of the domain in which the limiting process is considered.
Even for elementary functions, different choices of the domain may lead to different asymptotic interpretations.

1 The Function

Consider the real-valued function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

This function is continuous at $x = 0$, but it oscillates infinitely often in any punctured neighborhood of the origin.

2 Naive Asymptotic Question

A natural question is:

> Can the constant 0 be interpreted as an asymptotic of $f(x)$ as $x \to 0$?

The answer depends on what is meant by "$x \to 0$", that is, on the set of values of x that are allowed in the limiting process.

3 Derivative and Asymptotic

A similar ambiguity appears for the derivative. The derivative of f does not have a limit at $x = 0$, and therefore 0 cannot be interpreted as an asymptotic of $f'(x)$ in any neighborhood of the origin on the real axis. This already indicates that asymptotic behavior cannot be characterized solely by local expressions; the admissible range of the variable is essential.

4 Complex Extension

Extend the function to the complex plane by

$$f(z) = z^2 \sin\left(\frac{1}{z}\right), \qquad z \in \mathbb{C},\ z \neq 0,$$

with $f(0) = 0$.
In the complex plane, the question of asymptotic behavior necessarily involves the choice of a domain in which the limit is taken.

5 Restricted Domain

Consider the domain

$$D = \{\, z \in \mathbb{C} : \Im(z) < \Re(z)^2 \,\}.$$

Within this domain, the function $f(z)$ satisfies

$$\lim_{z \to 0,\ z \in D} \frac{f(z)}{z^2} = 0.$$

In this restricted sense, the constant 0 can be interpreted as an asymptotic of $f(z)$ at $z \to 0$ within the domain D.

6 Interpretation

This example demonstrates that:

- An asymptotic relation requires the specification of a domain.
- Without such a restriction, the notion of asymptotic behavior may become ambiguous or meaningless.
- Even elementary functions may admit different asymptotic interpretations in different domains.

The figure below illustrates the chosen domain and the behavior of the function in the complex plane.

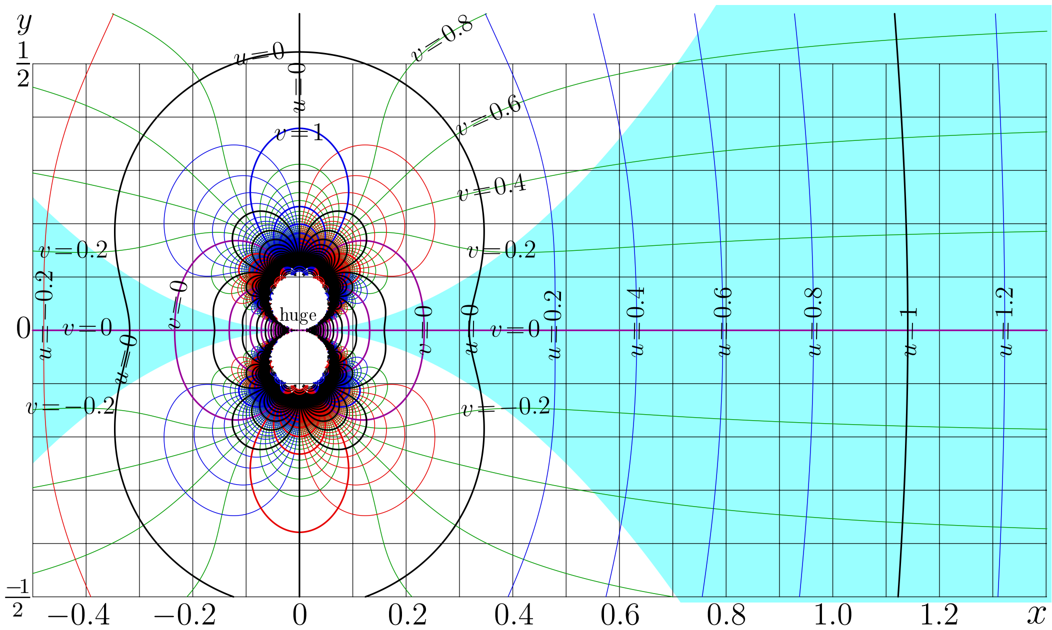

Fig. 4.1: Example of the mapping $u + \mathrm{i}v = f(x + \mathrm{i}y)$, where $f(z) = z^2 \sin(1/z)$. The shaded region corresponds to the domain $y < x^2$, in which 0 can be interpreted as an asymptotic of $f(z)$ at $z \to 0$.

Chapter 5

Stirling Asymptotics

1 Factorial and Gamma Function

The factorial function and the Gamma function are related by the identity

$$\Gamma(z+1) = z!.$$

This relation is understood as a definition of the factorial for non-integer values via the Gamma function.

2 Logarithmic Gamma Function

For complex z, LoGamma(z) denotes the principal branch of $\log \Gamma(z)$. The branch cut is taken along the negative real axis. The choice of branch affects asymptotic representations near this axis.

3 Stirling Expansion

For large $|z|$, the Gamma function admits an asymptotic expansion of the form

$$\Gamma(z) \sim \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} \sum_{k=0}^{\infty} \frac{g_k}{z^k},$$

as $z \to \infty$ within a sector that excludes the negative real axis.
This expansion is an asymptotic approximation in the sense defined in Chapter 2.

4 Domain of Validity

The Stirling expansion is not uniform with respect to the direction of approach to infinity.

In particular, the approximation deteriorates near the negative real axis, where different asymptotic representations are required.
Therefore, the specification of the domain is an essential part of the asymptotic statement.

5 Agreement Function

To quantify the quality of an approximation, the agreement function

$$a(z) = -\log_{10}\left(\frac{|F(z) - A(z)|}{|F(z)| + |A(z)|}\right)$$

is used.
The value of $a(z)$ provides an estimate of the number of correct decimal digits in the approximation $A(z)$ to the function $F(z)$.

Bibliography

[1] https://api.pageplace.de/preview/DT0400.9781439864548_A38306604/preview-9781439864548_A38306604.pdf Frank W.J. Olver. Asymptotics and Special Functions. Copyright O 1997by A K Peters, Ltd.

[2] https://dlmf.nist.gov/2#PT1 NIST Digital Library of Mathematical Functions. (observed in 2026) Chapter 2 Asymptotic Approximations by F. W. J. Olver and R. Wong

[3] https://www.cataloxy.ru/books/31903338_superfunktsii.htm https://mizugadro.mydns.jp/BOOK/202.pdf Дмитрий Кузнецов. Суперфункции. Lambert Academic Publishing, 2014

[4] https://www.morebooks.de/store/gb/book/superfunctions/isbn/978-620-2-67286-3 mizugadro.mydns.jp/BOOK/468.pdf Dmitrii Kouznetsov. Superfunctions. Lambert Academic Publishing, 2020

[5] http://photo-element.ru/offtopic/logic.html Беклемишев Дмитрий Владимирович. Заметки о женской логике. МФТИ, 2013.

[6] https://mizugadro.mydns.jp/t/index.php/Female_logic D.Beklemishev. Female logic.

[7] https://archive.org/details/bim_eighteenth-century_methodus-differentialis_stirling-james-frs-_1730 J. Stirling, *Methodus Differentialis*, 1730.

Not numbered links
https://en.wikipedia.org/wiki/Asymptotic_analysis