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Holomorphic asymptotics

Asymptotic expansions of holomorphic functions are considered. Examples are suggested and supplied with tables, explicit plots and complex maps.

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Preface

This book is devoted to the theory and computation of special functions, with emphasis on asymptotic methods and numerical evaluation. The material grew out of the TORI project.

Contents

Chapter 1

Prologue

This book is motivated by the attempts to load at server mizugadro the series of articles with definitions of terms related to the asymptotical analysis, «Asymptotic», «Sectorial asymptotic», «Strip Asymptotic». These concepts are imlicitely used in Mathematics and Physics at least since century 20, but the definitions found at the free access sites seem to be not sufficient. Among the sources available in the free access, the most complete seem to be the descriptions by Frank W.J. Olver and R. Wong [1, 2].
In book «Superfunctions» [3, 4], the concepts mentioned are used well beyond of the formal definitions. The euristic style of that book implies, that some definitions and deductions are omitted - over-vice, the book would be of size of "Abramotits-Stegun",
This book appears ad bunch of definitions; all other can be considered as illustrations. I try to follow the terminology from the publications cited, although the definitions are designed to be used for complex values of the arguments of the functions.
Some colleagues qualify many things as «obvious» - until some contradictions appear.
In the informal discussion, at some scientific and near-scientific slang, words «by definition» are used without referring to any definition of the term discussed. Practically, in such a slang, words «by definition» mean «I do not know and I do not want to know he definition of this term, but I want you to accept my statement without discussion.» Apparently, such an approach refers to the «motivated reasoning», so-called «Female logic» [5, 6].
I do not criticize such an approach. Perhaps, sometimes it is useful and even unavoidable in an euristical approach. But I still try to define the terms.

Chapter 2

Introduction

The goal of the scientific treat of any function; and, in particular, of a holomorphic function, is the conversion to a special function.
Definition.
Function f is qualified as «special function», if the following conditions are satisfied:
1. The definition of function f is available in the free access.
2. The properties of function f are investigated and available in the free access; in particular, the asymptotics.
3. Relations with other special functions are revealed.
4. The efficient algorithm of the evaluation is implemented and available in the free access.
Often, the efficient evaluation of a functions is performed through its asymptotic expansions.
In this book, I try to understand, what mathematical object can be qualified as an asymptotic of a function.

1 Naive approach

In the simplest approach we do not say anything about range of validity of the asymptotic; we just say, that the argument should have a huge value or a small value (or the difference between and argument and some constant should be a small value).
In this approach we assume, that, by default the argument of a function is element of some default set. Usually, it is set of real numbers or set of complex numbers.

Chapter 3

Asymptotic Objects

1 Function, Approximation, Residual

Let f be a function and A be another function. The difference

$$r(z) = f(z) - A(z)$$

is called the residual.

2 Asymptotic Relation

We say that A is an asymptotic of f as $z \to \infty$ in a domain $D \subset \mathbb{C}$ if

$$\lim_{|z|\to\infty,\ z\in D} \frac{r(z)}{f(z)} = 0.$$

3 Restricted Asymptotic

If the limit holds only in a restricted domain D, the asymptotic is called restricted.

4 Sectorial Asymptotic

If D can be chosen as a sector

$$D = \{z \in \mathbb{C} : |\arg z| < \alpha\},$$

the asymptotic is called sectorial.

5 Non-uniformity

The convergence above is not required to be uniform with respect to the sector parameter α. This non-uniformity is essential near singular directions.

6 Agreement as a Quantitative Measure

In numerical applications, the quality of an asymptotic approximation can be quantified using agreement functions, introduced in a later chapter.

Chapter 4

Example with sin

In order to show that term "asymptotic" is not so obvious, consider the example with elementary function:

$$f(x) = \begin{cases} x^2 \ \sin(1/x) & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases} \tag{4.1}$$

Can 0 be interpreted as asympototic of this function at zero?
Can 0 be interpreted as asympototic of derivative of this function at zero?
The answer depends on the set of values that are allowed for x.

Zero appears as asymptotic of function $f(z)$ at $z \rightarrow$ being considered at the range

$$\{z \in \mathbb{C} : \Im(z) < \Re(z)^2\} \tag{4.2}$$

The above seems to be simplest example, where, in order to get the asymptotic of the function, we beed to set some restriction on the range of values of its argument.

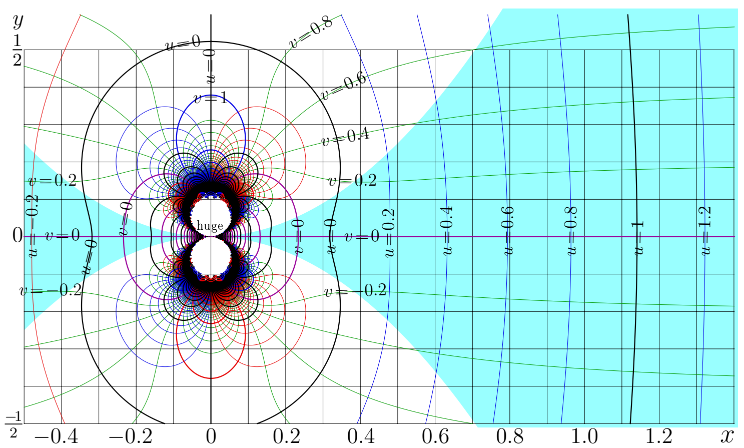

Fig. 4.1: Example $u + \mathrm{i}v = f(x+\mathrm{i}y)$; $y < x^2$ is shaded.

Chapter 5

Stirling Asymptotics

1 Factorial and Gamma Function

The factorial and the Gamma function are related by

$$\Gamma(z+1) = z!.$$

2 Logarithmic Gamma Function

For complex z, LoGamma(z) denotes the principal branch of $\log \Gamma(z)$.

3 Stirling Expansion

For large $|z|$ in a sector excluding the negative real axis,

$$\Gamma(z) \sim \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} \sum_{k=0}^{\infty} \frac{g_k}{z^k}.$$

4 Restricted and Sectorial Asymptotics

The asymptotic expansion above is sectorial and non-uniform near the negative real axis.

5 Agreement

The agreement function

$$a(z) = -\log_{10}\left(\frac{|F(z) - A(z)|}{|F(z)| + |A(z)|}\right)$$

estimates the number of significant decimal digits provided by the approximation A.

Bibliography

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[2] https://dlmf.nist.gov/2#PT1 NIST Digital Library of Mathematical Functions. (observed in 2026) Chapter 2 Asymptotic Approximations by F. W. J. Olver and R. Wong

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Not numbered links
https://en.wikipedia.org/wiki/Asymptotic_analysis