Proof of Kouznetsov's conjecture

James David Nixon
JmsNxn92@gmail.com

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Abstract

Suppose $T(z) : \mathbb{C} \to \mathbb{C}$ is an entire function. Suppose that $T : \mathbb{R} \to \mathbb{R}$ and is growing on the real-line. We provide a solution to the transfer equation $T(F(z)) = F(z+1)$ for $|\Im(z)| < \delta$ for some $\delta > 0$.

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1 Introduction

This paper is a brief proof of a conjecture of Dmitrii Kouznetsov. It requires we use infinite compositions; and a limiting process to solve an Abel equation at infinity. The author has written of infinite compositions in a total of 6 papers at this point. We won't use anything from these papers, except reference a theorem. We'll attach a proof of this theorem in the appendix.

Theorem 1.1. *Let $\{H_j(s,z)\}_{j=1}^\infty$ be a sequence of holomorphic functions such that $H_j(s,z) : \mathcal{S} \times \mathcal{G} \to \mathcal{G}$ where $\mathcal{S}$ and $\mathcal{G}$ are domains in $\mathbb{C}$. Suppose there exists some $A \in \mathcal{G}$, such for all compact sets $\mathcal{N} \subset \mathcal{G}$, the following sum converges,*

$$\sum_{j=1}^\infty ||H_j(s,z) - A||_{z\in\mathcal{N},s\in\mathcal{S}} = \sum_{j=1}^\infty \sup_{z\in\mathcal{N},s\in\mathcal{S}} |H_j(s,z) - A| < \infty$$

Then the expression,

$$H(s) = \lim_{n\to\infty} \Omega_{j=1}^{n} H_j(s,z) \bullet z = \lim_{n\to\infty} H_1(s, H_2(s, ...H_n(s,z)))$$

Converges uniformly for $s \in \mathcal{S}$ and $z \in \mathcal{N}$ as $n \to \infty$ to H, a holomorphic function in $s \in \mathcal{S}$, constant in z.

This theorem allows us to construct what we'll refer to as an asymptotic solution to the transfer equation. If we call a sequence of functions,

$$q_j(s,\lambda,z) = \frac{T(z)}{e^{\lambda(j-s)}+1}$$

Then these functions are holomorphic for $(s,\lambda) \in \mathbb{L} = \{(s,\lambda) \in \mathbb{C}^2 \,|\, \Re\lambda > 0,\ \lambda(j-s) \neq (2k+1)\pi i,\ j,k \in \mathbb{Z},\ j \geq 1\}$; and for $z \in \mathbb{C}$. Furthermore, for all compact sets $\mathcal{U} \subset \mathbb{L}$ and $\mathcal{K} \subset \mathbb{C}$ the following sum converges,

$$\sum_{j=1}^{\infty} ||q_j(s,\lambda,z)||_{\mathcal{U},\mathcal{K}} < \infty$$

Therefore, using Theorem 1.1, we can compose these as,

$$\rho_\lambda(s) = \Omega_{j=1}^{\infty} q_j(s,\lambda,z) \bullet z = \lim_{n\to\infty} q_1(s,\lambda,q_2(s,\lambda,...q_n(s,\lambda,z)))$$

Written more explicitly, this construction can be written,

$$\rho_\lambda(s) = \Omega_{j=1}^{\infty} \frac{T(z)}{e^{\lambda(j-s)}+1} \bullet z$$

This family of functions satisfies the functional equation,

$$\begin{aligned}
\rho_\lambda(s+1) &= \Omega_{j=1}^{\infty} \frac{T(z)}{e^{\lambda(j-s-1)}+1} \bullet z \\
&= \Omega_{j=0}^{\infty} \frac{T(z)}{e^{\lambda(j-s)}+1} \bullet z \text{ we've reindexed the composition from } j=0 \\
&= q_0(s,\lambda, \Omega_{j=1}^{\infty} \frac{T(z)}{e^{\lambda(j-s)}+1} \bullet z) \\
&= \frac{T(\rho_\lambda(s))}{e^{-\lambda s}+1}
\end{aligned}$$

Therefore, we are given the limiting property,

$$\frac{T(\rho_\lambda(s))}{\rho_\lambda(s+1)} \to 1 \text{ as } \Re(\lambda s) \to \infty$$

Where the error term drops off like $e^{-\lambda s}$. From this we can see that our function $\rho_\lambda(s)$ nearly approaches our desired solution $F(s)$ as $|s| \to \infty$. As we are interested only in the real line, we can restrict $\lambda \in \mathbb{R}^+$. We'll construct the function $F_\lambda(s)$ in the next section.

2 Constructing $F_\lambda(s)$

The function $T : \mathbb{R} \to \mathbb{R}$; and is a growing function. It is safe to assume that $T'(x) > 0$; as other wise the transfer equation can't be holomorphic on a strip including the real line. And from this we have some interval of the real-line in which $T : \mathbb{R} \to \mathcal{I}$ bijectively; where $\mathcal{I}$ may include the points at $\pm\infty$.

To begin, we note that,

$$\lim_{x\to-\infty} \rho_\lambda(x) = 0$$

And that for large enough X we know that for $x \geq X$ the values $\rho_\lambda(x) \in \mathcal{I}$. Define the sequence of functions,

$$F^n_\lambda(x) = T^{-n}(\rho_\lambda(x+n))$$

The job is to show this function is analytic as $n \to \infty$ on the strip $|\Im(s)| < \delta$ for some $\delta > 0$ depending on T. We provide a proof in the theorem below,

Theorem 2.1. *There exists a $\delta > 0$ and $L > 0$ in which $\lim_{n\to\infty} F^n_\lambda(s)$ converges uniformly on compact subsets of $|\Im s| < \delta$ while $\lambda > L$.*

Proof. We'll redefine out iteration by using a sequence of convergents $\tau^n_\lambda(s)$. We start these iterations as,

$$\tau^0_\lambda(s) = 0$$

And,

$$\tau^{n+1}_\lambda(s) = T^{-1}(\rho_\lambda(s+1) + \tau^n_\lambda(s+1)) - \rho_\lambda(s)$$

Then,

$$F^n_\lambda(s) = \rho_\lambda(s) + \tau^n_\lambda(s)$$

And we can think of $\tau^n_\lambda(s)$ as an error term between our transfer equation and the function ρ_λ. Taking differences, we get that,

$$\tau^{n+1}_\lambda(s) - \tau^n_\lambda(s) = T^{-1}(\rho_\lambda(s+1) + \tau^n_\lambda(s+1)) - T^{-1}(\rho_\lambda(s+1) + \tau^{n-1}_\lambda(s+1))$$

There exists a Lipschitz constant $A > 0$ associated with $\Re(s) > X$ and $|\Im(s)| < \delta$ such that,

$$|\tau^{n+1}_\lambda(s) - \tau^n_\lambda(s)| \leq A|\tau^n_\lambda(s+1) - \tau^{n-1}_\lambda(s+1)|$$

Iterating this procedure we arrive at,

$$|\tau^{n+1}_\lambda(s) - \tau^n_\lambda(s)| \leq A^n|T^{-1}(\rho_\lambda(s+n+1)) - \rho_\lambda(s+n)|$$

The term,

$$T^{-1}(\rho_\lambda(s+n+1)) = T^{-1}\left(\frac{T(\rho_\lambda(s+n))}{e^{-\lambda(s+n)}+1}\right)$$

Choose λ such that $q = e^{-\lambda}A < 1$. Then, using a Taylor expansion,

$$|T^{-1}(\rho_\lambda(s+n+1)) - \rho_\lambda(s+n)| \leq Ce^{-\lambda n}$$

For some $C > 0$. Therefore,

$$||\tau_\lambda^{n+1}(s) - \tau_\lambda^n(s)||_{\Re(s)\geq X,\, |\Im(s)|<\delta} \leq Cq^n$$

Choose $n, m > N$ such that,

$$\sum_{j=n}^{m} q^j < \epsilon/C$$

Then,

$$||\tau_\lambda^m - \tau_\lambda^n|| \leq \sum_{j=n}^{m-1} ||\tau_\lambda^{j+1} - \tau_\lambda^j|| < \epsilon$$

Therefore we have uniform convergence of,

$$F_\lambda^n(s) \to F_\lambda(s) \text{ as } n \to \infty \text{ for } \Re(s) \geq X,\, |\Im(s)| < \delta$$

This definition can be extended to $\Im(s) < \delta$ for small enough $\delta > 0$ by iterating T^{-1}. □

Et voilà; we've effectively proven Dmitrii Kouznetsov's conjecture.

3 Appendix

We've attached here a proof of Theorem 1.1.

Theorem 3.1. *Let $\{H_j(s,z)\}_{j=1}^\infty$ be a sequence of holomorphic functions such that $H_j(s,z) : \mathcal{S}\times\mathcal{G} \to \mathcal{G}$ where $\mathcal{S}$ and $\mathcal{G}$ are domains in $\mathbb{C}$. Suppose there exists some $A \in \mathcal{G}$, such for all compact sets $\mathcal{N} \subset \mathcal{G}$, the following sum converges,*

$$\sum_{j=1}^\infty ||H_j(s,z) - A||_{z\in\mathcal{N}, s\in\mathcal{S}} = \sum_{j=1}^\infty \sup_{z\in\mathcal{N}, s\in\mathcal{S}} |H_j(s,z) - A| < \infty$$

Then the expression,

$$H(s) = \lim_{n\to\infty} \mathop{\Omega}_{j=1}^{n} H_j(s,z) \bullet z = \lim_{n\to\infty} H_1(s, H_2(s, ...H_n(s,z)))$$

Converges uniformly for $s \in \mathcal{S}$ and $z \in \mathcal{N}$ as $n \to \infty$ to H, a holomorphic function in $s \in \mathcal{S}$, constant in z.

Proof. The first thing we show is for all $\epsilon > 0$, there exists some N, such when $m \geq n > N$,

$$|\underset{j=n}{\overset{m}{\Omega}} H_j(s,z) \bullet z - A| < \epsilon$$

For z in $\mathcal{N} \subset \mathcal{G}$ (where A is in the open component of $\mathcal{N}$), and $s \in \mathcal{S}$. This then implies as we let $m \to \infty$, the tail of the infinite composition stays bounded. Forthwith, the infinite composition becomes a normal family, and proving convergence becomes simpler. We provide a quick proof of this inequality.

Set $||H_j(s,z) - A||_{\mathcal{S},\mathcal{N}} = \rho_j$. Pick $\epsilon > 0$, and choose N large enough so when $n > N$,

$$\rho_n < \epsilon$$

Denote: $\phi_{nm}(s,z) = \Omega_{j=n}^m H_j(s,z) \bullet z = H_n(s, H_{n+1}(s, ...H_m(s,z)))$. We go by induction on the difference $m - n = k$. When $k = 0$ then,

$$||\phi_{nn}(s,z) - A||_{\mathcal{S},\mathcal{N}} = ||H_n(s,z) - A||_{\mathcal{S},\mathcal{N}} = \rho_n < \epsilon$$

Assume the result holds for $m - n < k$, we show it holds for $m - n = k$. Observe,

$$\begin{aligned} ||\phi_{nm}(s,z) - A||_{\mathcal{S},\mathcal{N}} &= ||H_n(s, \phi_{(n+1)m}(s,z)) - A||_{\mathcal{S},\mathcal{N}} \\ &\leq ||H_n(s,z) - A||_{\mathcal{S},\mathcal{N}} \\ &= \rho_n < \epsilon \end{aligned}$$

Which follows by the induction hypothesis because $\phi_{(n+1)m}(s,z) \subset \mathcal{N}$–it's in a neighborhood of A which is in $\mathcal{N}$. That is $m - n - 1 < k$.

The next step is to observe that $\Omega_{j=1}^m H_j(s,z)$ is a normal family as $m \to \infty$, for $z \in \mathcal{N}$ and $s \in \mathcal{S}$. This follows because the tail of this composition is bounded. We can say $||\,\Omega_{j=1}^m H_j(s,z)||_{\mathcal{S},\mathcal{N}} < M$ for all m.

Since $\phi_m(s,z) = \Omega_{j=1}^m H_j(s,z) \bullet z$ are a normal family for all compact sets $\mathcal{N} \subset \mathcal{G}$; there is some constant $M \in \mathbb{R}^+$ and $L \in \mathbb{R}^+$ such,

$$||\frac{d^k}{dz^k}\phi_m(s,z)||_{\mathcal{S},\mathcal{N}} \leq M \cdot k! \cdot L^k$$

To see this, take $|z - A| < 2\delta$ and observe,

$$\frac{d^k}{dz^k}\phi_m(s,z) = \frac{k!}{2\pi i}\int_{|\xi - A| = 2\delta} \frac{\phi_m(s,\xi)}{(\xi - z)^{k+1}}\,d\xi$$

So that, taking the supremum norm across $|z - A| \leq \delta$

$$\begin{aligned} ||\frac{d^k}{dz^k}\phi_m(s,z)||_{\mathcal{S},|z-A|\leq\delta} &\leq \frac{k!}{2\pi}\int_{|\xi - A| = 2\delta} \frac{||\phi_m(s,\xi)||_{\mathcal{S}}}{|\xi - z|^{k+1}_{|z-A|\leq\delta}}\, d\xi \\ &\leq \frac{k!}{2\pi}\int_{|\xi - A| = 2\delta} \frac{M}{\delta^{k+1}}\, d\xi \\ &\leq \frac{2Mk!}{\delta^k} \end{aligned}$$

Where we've used the bound $|\xi - z| \geq \delta$ when $|\xi - A| = 2\delta$ and $|z - A| \leq \delta$. This bound can be derived regardless of $\mathcal{N}$ for varying M and L.

Secondly, using Taylor's theorem,

$$\begin{aligned} \phi_{m+1}(s,z) - \phi_m(s,z) &= \phi_m(s, H_{m+1}(s,z)) - \phi_m(s,z) \\ &= \sum_{k=1}^{\infty} \frac{d^k}{dz^k}\phi_m(s,z)\frac{(H_{m+1}(s,z) - z)^k}{k!} \\ &= (H_{m+1}(s,z) - z)\sum_{k=1}^{\infty} \frac{d^k}{dz^k}\phi_m(s,z)\frac{(H_{m+1}(s,z) - z)^{k-1}}{k!} \end{aligned}$$

So that, setting $z = A$,

$$\begin{aligned} ||\phi_{m+1}(s,A) - \phi_m(s,A)||_{s\in\mathcal{S}} &\leq ||H_{m+1}(s,A) - A||_{s\in\mathcal{S}}\sum_{k=1}^{\infty} ML^k ||H_{m+1}(s,A) - A||^{k-1} \\ &\leq ||H_{m+1}(s,A) - A||_{\mathcal{S}}\frac{ML}{1-q} \end{aligned}$$

For $L||H_{m+1}(s,A) - A||_{\mathcal{S}} \leq q < 1$, which is true for large enough $m > N$. Setting $C = \frac{ML}{1-q}$. Applying from here,

$$||\phi_{m+1}(s,A) - \phi_m(s,A)||_{s\in\mathcal{S}} \leq C||H_{m+1}(s,A) - A||_{s\in\mathcal{S}}$$

This is a convergent series per our assumption. Choose N large enough, so that when $m, n > N$,

$$\sum_{j=n}^{m-1} ||H_{j+1}(s,A) - A||_{s\in\mathcal{S}} < \frac{\epsilon}{C}$$

Then,

$$\begin{aligned} ||\phi_m(s,A) - \phi_n(s,A)||_{s\in\mathcal{S}} &\leq \sum_{j=n}^{m-1} ||\phi_{j+1}(s,A) - \phi_j(s,A)||_{s\in\mathcal{S}} \\ &\leq C\sum_{j=n}^{m-1} ||H_{j+1}(s,A) - A||_{s\in\mathcal{S}} \\ &< \epsilon \end{aligned}$$

So we can see $\phi_m(s)$ must be uniformly convergent for $s \in \mathcal{S}$, and therefore defines a holomorphic function $H(s)$ as $m \to \infty$.

This tells us,

$$H(s) = \mathop{\Omega}_{j=1}^{\infty} H_j(s,z) \bullet z\Big|_{z=A}$$

Converges and is holomorphic. To show this function equals,

$$\mathop{\Omega}_{j=1}^{\infty} H_j(s,z) \bullet z$$

For all $z \in \mathcal{G}$; simply notice that,

$$\mathop{\Omega}_{j=m}^{\infty} H_j(s,z) \bullet z$$

Is arbitrarily close to A as we let m grow (which was shown at the beginning of this proof). Then,

$$\begin{aligned} \mathop{\Omega}_{j=1}^{\infty} H_j(s,z) \bullet z &= \mathop{\Omega}_{j=1}^{m-1} H_j(s,z) \bullet \mathop{\Omega}_{j=m}^{\infty} H_j(s,z) \bullet z \\ &= \lim_{m\to\infty} \mathop{\Omega}_{j=1}^{m-1} H_j(s,z) \bullet \lim_{m\to\infty} \mathop{\Omega}_{j=m}^{\infty} H_j(s,z) \bullet z \\ &= \mathop{\Omega}_{j=1}^{\infty} H_j(s,z) \bullet z\Big|_{z=A} \end{aligned}$$

This concludes our proof.

□

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