Math 1. (compiled 2021.09.16)

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Learn math

Abstract. Basic math tools are introduced.
Examples for teaching since "zero" are suggested (Fig.1).
Exercises for application of the tools are supplied.

Lesson 00.01. Preface

Here we present the first volume of manual of Math, adopted for students of the Galia National University of Tartaria. The use at other schools is also encouraged.

This textbook is based on questions, that arise at learning Mathematics at all departments of the University. The critics should be greatly appreciated.

Most of courses of century 21 about Math begin with arithmetics. That is difficult for beginners, who yet have no any idea about logic.. Well, they have some common sense, and even practice in arguing, that is sometimes called "logic" [1], but that thing actually is rather motivated reasoning, kind of art, that has nothing to do with the Boolean algebra, nor with Math. In this curse, we correct that fault.

The previous courses follow tradition of century 19, pre-computer era. That time, the books were difficult to get; one had to go to the shop, or to the library; manually find the book necessary, and then, either to buy it, or to borrow it, and to pay its price; it could take several hours. The authors tried to collect all the information, necessary for the course, in the same textbook.

In centuries 20 and 21, the referencing had been simplified, bit still many texts required registration of the reader, and the complicated procedure of the access had been required; sometimes just to understand, that the source is not that the reader is looking for.

According to Constitution of Tartaria, all the information here is free. So, do not repeat all the details the student need, but give a hint: what is really necessary to understand and to use Math; to avoid questions in style *"What scissors are better to hammer a screw into a wall?"*. However, if you still have a questions of such a kind, and the databases fail to explain, what do you indeed need, do not hesitate to ask; our professors are paid to answer all questions, including the stupid ones.

Asking questions, try to understand also, why did not you find the answer by yourself. These questions also can be addressed to teachers. Ability to formulate well your need and your questions is important tool, and our staff try to provide you with this tool.

The topic of this textbook can be illustrated with the old fairy tale "Crazy tailor" [2]:

One-legged friend of one Taylor asked him to sew the special one-leg pants.
He payed well for the custom pants, ans liked the result, but he needed also the pants for his dog, who, as himself, had lost one leg long time ago.Taylor sewed the pants for that 3-leg dog. The pants were beautiful, and friend of friend asked him the same for his normal, 4-leg dog..
The story is long, the starfishes and octopuses are mentioned there. En fin, the Taylor had elaborated tools to sew pantaloons for creatures with arbitrary number n of legs. And if tomorrow some extraterrestrials with n legs come, the Taylor already has pantaloons for them. ..

Hope, with the story above, the readers understand, what is Math about.

Contents

Chapter 01: Introfuction

Mathematic appears as the special language, designed to describe problems in a way, that simplifies their solution. This language is based on definitions, axioms, conjectures and theorems. Here, we begin to learn this language.

Lesson 01.01. Basic concepts

Often, mathematic refers to calculation of some quantities. The goal of this introduction is to mention wide variety of various kinds of numbers exist. There exist also many other objects the mathematicians deal with. They construct any object as soon as they consider it interesting, important or useful, and give it some name, making system of notations.

Historically, mathematic was about numbers and geometric figures. However, mathematic deal not only with numbers and geometric figures, but with any object(s), that any mathematician considers to be useful in formulation problems and the solution.

Mathematic appears as soon as some phenomena are qualified as similar. In order to apply mathematic to some thing, it should be considered as an "object". Mathematicians and students, who deal with objects, are qualified as subjects. Some objects, with permission of mathematicians, can be qualified also as subjects, as they, by themselves, deal with objects.

Any mathematical object should be supplied with a description. definition.
Each object is supposed to belong to some set.

Mathematicians deal with constants and variables.
The simplest variables are logical ones. They quality statements in terms of "true" and "false". They form base of logic. Then, other mathematical objects can be constructed:
From the logical variables, one construct natural numbers, as sequence of the logical variables.
From a pair of natural numbers, one constructs the integer numbers.
From a pair of integer numbers, one constructs the rational numbers.
Sequences of the rational numbers are considered as real numbers.
Then, sequences of real numbers are considered as vectors.
Sequences of vectors are considered as matrices, and so on.

Mathematicians construct many other objects; usually, they are built-up from the objects, that already had been defined.

In such a way, mathematicians deal with constructions. Then, mathematicians deal with axioms, conjectures and theorems about their constructions.

In order to deal with some object, one needs to define, whit set does it belong. The description of the set implies some axioms, that are the object is supposed to satisfy. The constructions, definitions, axioms, conjectures and theorems form a scientific concept.

Lesson 01.02. Mathematics and other sciences

We need to indicate place of mathematic among other sciences. The concept is called “scientific”, if it all the 6 properties below:

1. Applicability: The concept has limited range of validity, distinguishable from Empty Set.

2. Verifiability: In the terms of the already accepted concepts, some specific experiment with some specific result, that confirms the concept, can be described.

3. Refutability: In the terms of the concept, some specific experiment with some specific result, that negates the concept, can be described.

4. Self-consistency: No internal contradictions of the concept are known.

5. Principle of correspondence: If the range of validity of a new concept overlaps the range of validity of another, already accepted concept, then, the new concept either reproduces the results of the old concept, or indicates the way to refute it. (For example, the estimate of the range of validity of the old concept may be wrong.)

6. Pluralism: Mutually-contradicting concepts coexist; if two concepts, satisfying the 1-5 above, have some common range of validity, then, in this range, the simplest of them has priority.

In such a way, Mathematics is not an exception; it is considered as science, as all other sciences.

The first, most difficult and most important part at learning of mathematics is to understand, which mathematical constructions will be requested in life, in order to learn namely these constructions. Then, one can find descriptions of the topics required and, if difficulties, ask the Teacher.

Second, equally important part is to determine, in which order the topics can be learned. For example, one cannot deal with multiplication before to learn summation; one cannot deal with derivatives before to practice with functions.

Often, it is assumed, that the teachers or ministers of education know mathematics well enough, to determine, what knowledges will be requested, and in which order these knowledges should be given to students.

However, this does not apply to Tartaria, as the people capable to logic, had been eliminated since century 20, since the first “Philosopher ships”, that carried away from the country the smartest artists, researchers and engineers. Then, during almost two centuries, the education has been substituted with propaganda. The students were forced to get not the knowledges they need in the life, but the doctrines, that bring profit to the usurpers and other offees.

The people of our country have genetic tendency to autocracy, despotism. Despotism is supported by the lack of knowledge. In order to resist against this tendency, we need to learn to determine, which knowledge is necessary, how to get it and how to use it. Math gives the base knowledge; it allows systematization of all other knowledge.

The most important part of mathematic is logic. It makes difference between Science and *Motivated reasoning*. The last is called also *Female logic* [1], it may be also of interest, but it falls out of scope of this manual. So, we begin with logic.

Chapter 02: Logic

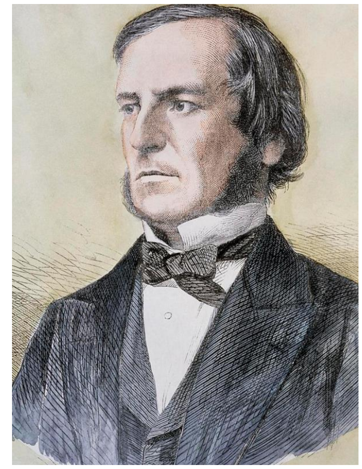


Fig. 02.01: George Bool

There are two basic styles of arguing, the Scientific discussion and the Motivated reasoning [1].

The scientific discussion is based on the logic (*Mathematical logic*), its goal is deduction of new things and revealing of errors.

The logic, in the mathematical sense, is attributed to Aristoteles and George Bool shown in Fig.02.01. This logic is introduced below.

Lesson 02.01. Logical constants and variables

Logical variables are usually assumed to have only one of two possible values, "true" or "false". This is simplest example of the mathematical abstraction. Often, some statements in our life can be qualified in terms of "true" and "false".

Exercise: Suggest few examples of true statement and few examples of false statements.

In order to simplify formulas, let F denote "false", and let T denote "true".

In mathematics, the constants are usually denoted with Roman letters. The examples are letters F and T, used here in the specific meaning above. Constants are not supposed to change their meaning.

Other Mathematical objects may change their meaning; then they are called "variables". Variables often are denoted with Italics letters.

If one have formulated some hypothesis, one can denote is with such a letter. Examples:

(02.01) a = No Human can daily eat food in amount that exceeds his weight.

(02.02) b = The barber shaves everyone who does not shave himself, but no one else.[3]

(02.03) c = Bellman caught the Snark.[4]

(02.04) d = God is never wrong.[5]

(02.05) e = The sky is blue.

Fig. 02.02: The sky is blue

Such a notation does not yet specify, wether the hypothesis a, b, c, d, e are T (true) of F (false). It does not even specify, either any of values F of T can be assigned to each of these hypothesis. Perhaps, some of these statements are valid within some paradigms, and invalid at other. According to Axiom 1, every scientific concept has limited range of validity.

In some systems of notations, some hypothesis may have mathematical sense; then they can be qualified as "false" or "true". Namely for these cases, we learn to operate with various notations and various hypotheses. For these cases, we give the hypotheses names and numerate the statements, at it is shown in equations (02.01-(02.05).

Lesson 02.02. Logical "and" and logical "or"

Logical operations are defined on the logical objects, constants and/or variables.

Ability to generalize observations in terms of mathematical hypothesis is most difficult part of the application of mathematics. And even if the primary part is performed, we need to learn to operate with statements.

The basic mathematical operation is assignment. It is denoted with character "=".

If, within some consideration, two objects are equivalent, it is denoted with symbol "==". If some logical value a had been assigned to variables b and c, then they are equivalent and we say, that

(02.06) $\quad b == c$

This is basic property of mathematical objects; if the two variables are assigned the same value, his mean, that they are equivalent, and there is no mathematical way to distinguish them. If it causes no confusion, symbol = is also used to denote equivalence of two objects. In all cases, when equivalence can be confused with assignment, the equivalence should be written as ==.

Define operation of logical addition (logical or) $\vee$ in the following way:

(02.07) $\quad \mathrm{T} \vee \mathrm{T} = \mathrm{T}$

(02.08) $\quad \mathrm{T} \vee \mathrm{F} = \mathrm{T}$

(02.09) $\quad \mathrm{F} \vee \mathrm{T} = \mathrm{T}$

(02.10) $\quad \mathrm{F} \vee \mathrm{F} = \mathrm{F}$

Define operation of logical multiplication (logical "and") $\wedge$ in the following way:

(02.11) $\quad \mathrm{T} \wedge \mathrm{T} = \mathrm{T}$

(02.12) $\quad \mathrm{T} \wedge \mathrm{F} = \mathrm{F}$

(02.13) $\quad \mathrm{F} \wedge \mathrm{T} = \mathrm{F}$

(02.14) $\quad \mathrm{F} \wedge \mathrm{F} = \mathrm{F}$

For any logical operation $\mathcal{L}$, the following relation takes place:

(02.15) $\quad a\mathcal{L}b = b\mathcal{L}a$

Here, symbol $\mathcal{L}$ denotes any of operations $\wedge$ or $\vee$; this can be written as follows:

(02.16) $\quad \mathcal{L} = \wedge \;\text{ or }\; \mathcal{L} = \vee$

Property (02.15) is called commutativity. The logical operations are commutative. In order to help to avoid confusions with logical variables, the logical operation is denoted with calligraphic letter.

Excercice. Check relation (02.15) for $\mathcal{L} = \wedge$ for all possible values of a and b, using definition (02.07)-02.10).

Excercice. Check relation (02.15) for $\mathcal{L} = \vee$ for all possible values of a and b, using definition (02.11)-02.14),

Note for future: Relation, similar to (02.15), takes place for many operations, including summation and multiplication of numbers, considered in the next chapter.

Lesson 02.03. Notations and characters

In principle, any text can be represented in binary form, using the only two characters. Also, in principle, for any new object or subject, the new special characters could be invented. Here, we try to find some compromise, keeping some reasonable amount of characters to describe mathematic.

We use Roman Latin characters for the main text and for constants (although, the Italics can be used for emphazing of some words).

Exersice: Write all characters of the Roman Latin alphabet.

We use Italics Latin characters for variables.

Exercise: Write all characters of the Roman Italics alphabet. Be sure, that the Italics characters are not confused with the Roman characters.

We use calligraphic capital letters to denote operations.

Exercise: Write all capitals characters of the Roman calligraphic alphabet.
Be sure that they differ from the Italics letters:

$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$

We need also the special characters $\wedge$ and $\vee$ for logical operation on logical operands.

Exercise: Write several times $\vee$ and letter “v” in such a way, that shows the difference in writing of these characters.

Also, some Greek characters are used in scientific notations. We'll need characters

α, β, Γ, γ, Δ, δ, ϵ, ε, η, κ, $\varkappa$, χ, μ, ν, Φ, ϕ, φ, Ψ, ψ, Θ, θ, ϑ, Π, π, ρ, ϱ, Σ, σ τ, Ξ, ξ, ζ, Ω, ω

In Greek alphabet, there exist also other characters, but they are not so often in the mathematical notations.

Exersice. Write the Greek letters from the list above in such a way, that they are difficult to not confuse with each other, nor with the Latin characters.

There exist also some other special mathematical symbols.
Two of them are already introduced: $\wedge$ and $\vee$.

Be careful that in your writing, character $\vee$ is not confused with letter V, nor v.

Be careful that in your writing, character $\wedge$ is not confused with letter Λ.

Then, un future chapters, we need even more characters; especially, the followings:

$$+, -, \times, *, /, \sum, \prod, \int$$

Keep in mind, that, in your writing,
character $\times$ should not be confused with Latin letter x,
Greek letter χ ishould not be confused with Italics Latin letter X,
Greek letter γ should not be confused with Italics Latin letter r,
Greek letter η should not be confused with Italics Latin letter n,
Greek letter ν should not beconfused with Italics Latin letter v,
Greek letter ρ should not be confused with Italics Latin letter p.

Also, we need parenthesis (), they indicate the order of mathematical operations.
In the next lesson, we begin to use characters, introduced above.

Lesson 02.04. Parenthesis and Associativity

Consider expression

(02.17) $(a\mathcal{L}b)\mathcal{M}c$

Here, the parenthesis () indicate, that first, operation $\mathcal{L}$ is performed for operands a and b; and then, the result of this operation appears as first operand of operation $\mathcal{M}$.

Similarly, in writing

(02.18) $a\mathcal{L}(b\mathcal{M}c)$

the parenthesis () indicate, that first, operation $\mathcal{M}$ is performed with arguments b and c; then, the result is used as second operand for operation $\mathcal{L}$.

For any logical operation $\mathcal{L}$, id est, $\mathcal{L}=\wedge$ or $\mathcal{L}=\vee$, the following relation takes place:

(02.19) $(a\mathcal{L}b)\mathcal{L}c = a\mathcal{L}(b\mathcal{L}c)$

This property is called *associativity.* Each of logical operations $\vee$ and $\wedge$ is associative. Such a property looks intuitive and obvious. For this reason, it is good example to learn substitute various quantities to the already written formulas. The associativity allows to write the logical expression with three operands without parenthesis:

(02.20) $a\mathcal{L}b\mathcal{L}c = (a\mathcal{L}b)\mathcal{L}c = a\mathcal{L}(b\mathcal{L}c)$

Exercise: Check the associativity for some of possible meanings of operator $\mathcal{L}$ and some values of the operands a, b, c. Rewrite formula (02.20) for $\mathcal{L} = \wedge$ and for $\mathcal{L} = \vee$.

Exersice: Consider expression

(02.21) $a\mathcal{L}b\mathcal{L}c\mathcal{L}d$

Practice in correct ways to add parenthesis in expression (02.21).

Exersice: Rewrite (02.21), replacing $\mathcal{L}$ to any of its possible meanings; for example, substitute $\mathcal{L}$ to $\vee$.

Exersice: Rewrite it(02.21), replacing some of variables a, b, c, d to constants T or F, by your choice. Check the result, using definition of operations $\vee$ and $\wedge$.

In the expressions above, it is important, that in each formula, all occurrences of $\mathcal{L}$ are replaced to the same operation. Over-vice, the formula may happen to be non-correct.

Exersice: Consider expression

(02.22) $(a\mathcal{L}b)\mathcal{M}c$

Compare it to expression

(02.23) $a\mathcal{L}(b\mathcal{M}c)$

for $\mathcal{L}=\wedge$, $\mathcal{M}=\vee$ and various examples of a, b, c. Check that, for some of a, b, c, expressions (02.22) and (02.23) give different values.

Lesson 02.05. Distributivity

The last exercise from the previous section indicate that the logical expressions may be not so trivial. For any logical operations $\mathcal{L}$ and $\mathcal{M}$, the following relations takes place:

(02.24) $$(a\mathcal{L}b)\mathcal{M}c \;=\; (a\mathcal{M}c)\mathcal{L}(b\mathcal{M}c)$$

As we do not specify here meaning of operation $\mathcal{L}$ and $\mathcal{M}$, the same can be written as

(02.25) $$(a\mathcal{M}b)\mathcal{L}c \;=\; (a\mathcal{L}c)\mathcal{M}(b\mathcal{L}c)$$

In some systems of notations, character $+$ is used instead of $\vee$; as for $\wedge$, it is omitted. then, the formula or distributivity can be written as follows

(02.26) $$a(b+c) = ab + ac$$
(02.27) $$(a+b)c = ac + bc$$

Similar formulas take place also for numbers.

Exercise. Check, that equation (02.27) follows from (02.26) and the commutativity of operation with omitted notation and operation $+$.
Hint: change names of the dummy variables; for example replace
$a \mapsto x$
$b \mapsto y$
$c \mapsto z$
in one formula and
$a \mapsto z$
$b \mapsto y$
$c \mapsto x$
in another one.

About notations. The new mathematical character "$\mapsto$" is introduced above . Writing

$p \mapsto q$

means, that in the expression, letter p should be replaced with letter q. However, the objects p and q are supposed to be from the same set, allowed in the context.

Exercise: Rewrite equation (02.24) for various values of logical variables a, b, c. Try to find at least one case, when any (02.24) or (02.25) fails.

Note for future. The same relation (02.26) keeps also for other meaning of symbols involved. For example, in the next chapter, the Italics Roman letters denote numbers, character $+$ may denote addition of numbers, and omitted character * may denote multiplication of numbers, id est, the arithmetical operations. For arithmetical operations, the meaning of parenthesis remain the same; the parenthesis indicate order of operations at the evaluation of the expression.

Lesson 02.06. Exercices

Exercise: Simplify logic espression $x = (a \vee b) \vee a$

Solution:
Using the commutativity, we get
$x = (b \vee a) \vee a$
Using associativity, we get
$x = b \vee (a \vee a)$
$a \vee a$ is always a
Hence, the initial expression can be written as $x = a \vee b$

Exercise: Simplify the logic expression $(a \wedge b \wedge c) \vee a$

Exercise: Simplify the logic expression $(a \wedge c) \vee a$

Exercise: Suggest more examples for simplification of the logical expressions introduced above. Search in the literature is encouraged; then, the source of the example should be attributed.

Lesson 02.07. Negation

Negation $\neg$ is logical operation with single argument. It means statement, opposite to its operand:

(02.28) $\neg\mathrm{T} = \mathrm{F}$
(02.29) $\neg\mathrm{F} = \mathrm{T}$

Example. For statement
$x =$ Fuhrer is good
the negation is
$y =$ Fuhrer is not good
The relation between x and y can be written as follows:
$x = \neg y$
The same can be written also as follows:
$y = \neg x$

Operation of negation has simple property: negation of negation of any logical statement x is just x: $\neg(\neg x) = x$

Exercise: Suggest more examples of negation of statements

Historical note: In century 20, during the USSR, the use of logic at schools was not welcomed: With singe symbol $\neg$, almost every statement of Soviet propaganda can be converted to antisoviet, that had been considered as a serious crime (anti-Soviet propaganda).

Lesson 02.08. Identities

The following relation holds for logical variables a and b:

(02.30) $\neg(a \vee b) = \neg a \wedge \neg b = (\neg a) \wedge (\neg b)$

There is agreement, that operation of negation has priority: it is performed prior to $\wedge$ and $\vee$.

(02.31) $\neg a \vee b = (\neg a) \vee b$

rather than $\neg(a \vee b)$

Exercise: check, that for any logical a, the relations hold:

(02.32) $a \vee \neg a = \mathrm{T}$

(02.33) $a \wedge \neg a = \mathrm{F}$

Exercise: Check, that for any logical a and b, the relations hols

(02.34) $\neg(a \vee b) = \neg a \wedge \neg b$

(02.35) $\neg(a \wedge b) = \neg a \vee \neg b$

Exercise: Suggest more logical identities, using binary operations $\wedge$, $\vee$ and unary operation $\neg$. Check validity of your formulas.

In such a way, we have considered the logical constants (F or T), logical variables and logical operations. Logical operations are important as they form the base of mathematics. In the next lesson, the examples of application of logical variables are considered.

Lesson 02.09. Secondary operations

The the previous lessons, the three logical operation are itroduced: Logical "and" $\wedge$, Logical "or" $\vee$ and negation $\neg$. Define three more logical operation with

(02.36) $x \rightarrow y \;=\; \neg x \wedge y$

(02.37) $x \oplus y \;=\; \neg(x \equiv y) = (x \wedge y) \vee (\neg x \wedge \neg y) = (x \wedge \neg y) \wedge (\neg x \vee y)$

(02.38) $x \equiv y \;=\; \neg(x \oplus y) = (x \wedge y) \wedge (\neg x \wedge \neg y)$

Exercise:
Construct table of values for each or operations $\rightarrow$, $\oplus$ and $\equiv$.

Exercise:
Suggest examples with simplification of expressions with operations $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\oplus$ and $\equiv$.

Lesson 02.10. Applications

Exercise.
In century 20, some elevators had no indivation of movement, up or down.
Professor of Logic goes up in such an elevator, and it stops to pick up one passenger.
The passenger asks:
– This elevator goes up or down?
– True.
– Do you go down?
– False.
– Do you go up?
– True.
Interpret the dialog in terms of mathematical logic.


Fig. 02.03: Are we in Bugiard?

Exercise. Gelsomino come country Gialand.
In this country, there are only two cities: Verdad and Bugiard.
All the people of the country live in these two cities.
Citizens of Verdad always say true.
Citizens of Bugiard always say lie.
Gelsomino enters one of two cities of Gialand.
He talks to one Customer in cafe, who is supposed to be either citizen of Verdad or citizen of Bugiard.
Gelsomino asks two questions:
A. Are we in Bugiard?
B. Do you live in this city?
Each of this questions is answered with either "Yes", that is interpreted as T, or "Not", that is interpreted as F. Denote answer to question A with letter a and answer to question B with letter b. Let
$x =$ Gelsomino is in Bugiard
$y =$ Gelsomino is in Verdad
Using logical operations, Express x and y in terms or a and b.

Hint. Consider four options below:
1. Gelsomino is in Bugiard, the Customer is from Bugiard.
2. Gelsomino is in Verdad, the Customer is from Bugiard.
3. Gelsomino is in Bugiard, the Customer is from Verdad.
4. Gelsomino is in Verdad, the Customer is from Verdad.

Emulate answer of the Customer in each of this case. Try to express x and y with simple formulas.
Solution: $x = \neg b$, $y = b$.
Note that the first question does not help Gelsomino to understand, which city is he visiting.

Exersice. How logic help to understand, that the superior power in country is usurped.
Here is historic example from century 20. Denote with letters a, b, c ,d the following statements:

(02.39) a = In a prosper country, shortage of food is not possible
(02.40) b = At the honest government, the country is prosper
(02.41) c = In the USSR, shortage of food is observed
(02.42) d = The government of the USSR is honest

Consider system of statements a, b, c, d. Can all these variables have value T?

Hint: $(a \wedge b \wedge c) \rightarrow \neg d$

Lesson 02.11. Logical arguing and motivated reasoning

Generally, in order to deduce something, using logic, some assumptions are necessary, some statements should be assumed. Not all systems of assumptions are valid.

Formalism or logic allows the special kind of discussion, namely, scientific discussion. In some cases, the participants of the discussion, opponents, agree to consider certain concept, based on certain initial statements, postulates. Then, from these statements, one can deduce other statements, check someone else's deduction. However, the subject of discussion should be formulated in terms of logical variables. Usually, it is assumed, that the system of logical rules is self-consistent. In addition, often, it is believed, that the system of initial assumptions is also self-consistent.

Sometimes, such a believe is based in sequences of tests, attempts to find an internal contradiction in the system of initial assumptions, if all these attempts fail. Then the logic can be applied.

Cases, when the logical arguing can be applied, are relatively rear. Often, the opponents cannot find any appropriate concept to use as common base for the deduction and the discussion. Then, they cannot construct logical expressions suitable to convince the opponent.

One can construct the "extended logic", adding two more constants,
NULL = Questionable and
Nonsense = Contradiction

with the following rules:

$$\text{T} \vee \text{NULL} = \text{T} \qquad (02.43)$$
$$\text{F} \vee \text{NULL} = \text{NULL} \qquad (02.44)$$
$$\text{F} \wedge \text{NULL} = \text{F} \qquad (02.45)$$
$$\text{T} \wedge \text{NULL} = \text{NULL} \qquad (02.46)$$
$$\neg\text{NULL} = \text{NULL} \qquad (02.47)$$
$$a \wedge \text{Nonsense} = \text{Nonsense} \qquad (02.48)$$
$$a \vee \text{Nonsense} = \text{Nonsense} \qquad (02.49)$$
$$\neg\text{Nonsense} = \text{Nonsense} \qquad (02.50)$$
$$\text{Nonsense} \wedge \text{NULL} = \text{Nonsense} \qquad (02.51)$$
$$\text{Nonsense} \vee \text{NULL} = \text{Nonsense} \qquad (02.52)$$

where a is any logical (not extended logical) variable.

Most of statements in the human life have value NULL or even Nonsense; at least for some of opponents. Usually, this disables the scientific discussion of habitual topics. So, the most of habitual discussion follow rules of motivated reasoning [1], not formal logic. From the point of view of the formal logic, such a discussion has no meaning since beginning, as the most of statements involved should be assigned non-logical values NULL or even Nonsense.

Example. Up to century 22, it is not clear, did Stalin dead by himself or he had been killed by his criminal partners. It is not yet even clear, is it possible to qualify his death in therms "dead by himself" or "had been killed". So, hypothesis "Stalin had been killed" can be evaluated as NULL. However, in some specific concepts, more specific value can be assigned to this hypothesis. [6]

Example: *Have you stopped to vodka every morning?* No one answer to this question can be treated as variable of formal logic; any answer to such a question happens to be wrong [7].

Questions, that do not allow answers, usable as logical variables, in mathematic are qualified as "incorrect". The popular error of young mathematicians is attempt to use mathematical logic in a discussion, where the motivated reasoning is used, instead of to say frankly: "your question is not correct" or "you use terms that are not well defined".

Lesson 02.12. Overview about logic

In this chapter, the concept of logical constants and logical variables is introduced.

The logical constants are denoted with letters F and T; for mnemonics an semantics, F is interpreted as False, while T is interreted as True.

The basic logical operations

$\wedge$, $\vee$, $\neg$

are defined with tables of values.

They have simple properties, that can be deduced from the definition.

The secondary logical operations

$\rightarrow$, $\oplus$, $\equiv$

are constructed from the basic logical operations.

Exersices

1. Found basic logical operations in the language you use to program your robots.
In that language, define secondary logical operations through the basic ones.
Do your definition coincide with those suggested by the authors, created that language?

2. Construct a table of values for each of logical operations mentioned above.

3. Suggest examples where someone, robot or human, has to deal with logical variables.

Not all practical situations can be described in terms of logic. The classical example is question

«Have you stopped drinking cognac in the mornings?» [7]

None of answers to this question can be treated as logical value. Such questions are qualified as non-correct; the suggestions of the answer cannot be treated as classical logical variables.

Excercice. Construct more non-correct questions, that sometimes do not allow an answer, interpretable as logical valie.

Example: Have you already paid the fine ticket for parking on the flower bed?

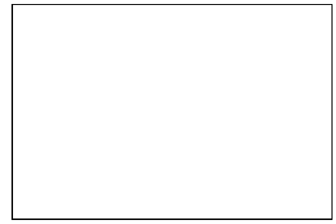
In practical life, often the motivated reasoning (sometimes denoted also with term "Female logic") [1] is used instead of logic. Motivated reasoning is style of arguing that look similar to logic, but operates with non-correct questions and statements; id est, statements, that cannot be considered as logical variables. Such kind of arguing falls out of scope of this manual.

Even at the scientific approach (that opposes the motivated arguing), the Formal logic suggested above has narrow range of applicability. In particular, logic in not efficient dealing with comparative, relative properties: "big", "huge", "small", "strong", "long", "short", "high11, "precise", "many", "much", "easy", "difficult", etc. In many cases, the comparison with some etalons still allows to use mathematics in these case, performing the quantitative analysis.

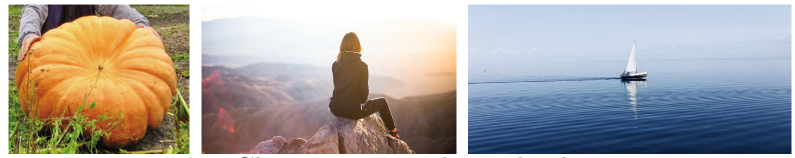
The logical variables, by themselves, are not efficient to deal with quantities. The more efficient objects are numbers. Numbers can be constructed from logic. This makes subject of the next chapter.

Chapter 03: Natural numbers

Historically, numbers seem to appear before logic. One can operate with numbers using some naive, intuitive understanding of term "logic". At least until century 21, in schools, children learn to deal with numbers first, before logic and sometimes without any logic. Due to extensive practice with arithmetics, sometimes, some students get also some skills operating logical constants and variables. But amount of wrong publications indicate, that many colleagues have primitive, naive concepts of the logical deduction [1]. We should not blame the majority of population: they are not guilty; at school, the teachers gave them knowledges about arithmetics and algebra; sometimes even mathematical analysis; but the most important element of mathematics often had been missed. Some examples from centuries 20 and 21 are collected [8]. In this course, we try to avoid this fault: here, numbers appear, when the fundamentals of logic are already declared in the previous chapter.



class 0: empty set



Class 1: sets with single element

Class 2: sets with two elements

Class 2: sets with three elements

Fig. 03.01: Arising of numbers

The idea of numbers arises from similarities of some sets. Let two sets be considered as equivalent, if each element of one set can be appointed with single element of another set. Such a class of equivalence appears as set of numbers.

However, being able to read the texts, the students already have some naive idea, what are numbers. In certain intuitive sense, one can reveal certain similarities in pictures, shown in each row of figure 03.01

At the zeroth row, there is only one picture. It is an attempt to draw the empty set.

Each image of the first row shows a single main object: single (although big) pumpkin, lonely girl in mounts, and the single yacht in the sea.

Each image of the second raw shows a couple of the main objects: couple of the cherry fruits, couple of humans.

Each image of the second raw shows a triad of objects: three fruits, three humans, three socks.

In certain sense, the sets, shown in each raw of the figure, are equivalent. This sense is amount of main objects shown. In order to show this equivalence in more accurate way, we need numbers.

Natural numbers appear as set of sets of objects with certain class of equivalence. Some sets are considered as equivalent. For some problems, it is not so important, what objects are under consideration, but it is important, how many of them are in each set. Namely for this cases, the numbers are invented.

Set of natural numbers often is denoted with symbol $\mathbb{N}$. Some times, the subscript "0" is added to this notation, in order to indicate, that the empty set is also included to this class

Many mathematicians believe, that there exist the special set of objects, this set is denoted with term "Natural numbers", with properties described below.

Lesson 03.01. Unity increment and zero

On the set of natural numbers, the operation "unity increment" is defined; and the result of this operation is also natural number. This operation can denoted with symbol ++. This can be written in the following way:

(03.01) $\forall n \in \mathbb{N}, \mathbin{++} n \in \mathbb{N}$

The following notations are used here.

Symbol $\in$ means, that the object at its left hand side is element of the set, specified at its right hand side. This symbol has easy mnemonics; it looks as letter E, written upside-down.

Symbol $\forall$ is read al “for all” or “for any”, dependently on the preferences of the Reader in the English grammar and semantics. This symbol also has easy mnemonics; it looks as upside-down letter A.

Formula (03.01) can be read in the following way:

For any n, that is element of set of natural numbers, operation ++ is defined, and $\mathbin{++} n$ is element of the same set.

In some programming languages, operator ++ have slightly different meaning; it applies the unity increment to its argument, changing its value (and for this reason cannot be applied to a constant); but in this course, ++ means just the result of application of this operation.
In some programming languages, operation ++ is not implemented. Instead of to write $\mathbin{++} a$, one has to type $(1+a)$, that, basically, corresponds to the common sense about this operation;

(03.02) $1 + n == \mathbin{++} n$

However, in the programming languages, operation ++ n has slightly different meaning, as it increments for unity value, stored in variable n.

Note, that ++ is operation with single argument, in the same way, as logical negation $\neg$ for logical variables and constants. Natural numbers are a little bit more complicated, that logical objects. In particular, for operation ++, there is no analogy of property

(03.03) $\neg\neg = \mathcal{I} =$ identity function

Here the identity function $\mathcal{I}$ is such function, that its result is the same as its argument.

Among natural numbers, there exist special initial element, denoted with character 0, such that, for any natural number $n \neq 0$,

(03.04) $(\mathbin{++} n == 0) = \mathrm{F}$

Such an element is unique; the only one natural number cannot be obtained with operation ++ from another natural number.

Lesson 03.02. Notations

$++\,2 = 3$

$++\,6 = 7$

Fig. 03.02: Applications of ++

The zeroth element of set of natural numbers, id est, 0, is used to define the single-character notations for some numbers:

(03.05) $1 = ++\,0 =$ one
(03.06) $2 = ++\,1 =$ two
(03.07) $3 = ++\,2 =$ three
(03.08) $4 = ++\,3 =$ four
(03.09) $5 = ++\,4 =$ five
(03.10) $6 = ++\,5 =$ six
(03.11) $7 = ++\,6 =$ seven
(03.12) $8 = ++\,7 =$ eight
(03.13) $9 = ++\,8 =$ nine

Number $++\,9$ is denoted with word "ten". This number has no unique, commonly-accepted single character name; sometimes, it is called X. These names are called cifras (the name is borrowed from Spanish), or, more officially, “Numerical digits”.

Examples of application of the notations are shown in Fig. 03.02. Formulas for notations 3 and 7 are shown. The illustrations are borrowed from the old publications by Papyrus [9] and [10]. In order to correspond better notations with prefix ++, both images are flipped right-left. In order to specify, that some object is some list is identified with some specific number, the following notations are used:

(03.14) 0th = zeroth
(03.15) 1st = first
(03.16) 2d = second
(03.17) 3rd = third
(03.18) 4th = fourth
(03.19) 5th = fifth
(03.20) 6th = sixth
(03.21) 7th = seventh
(03.22) 8th = eight
(03.23) 9th = ninth
(03.24) $(++\,9)$th = tenth

To memorize characters 0,1,2,3,4,5,6,7,8,9, the special font is suggested in Fig.03.03; the number of internal angles between lines corresponds to the natural number, denoted with the cifra. The font is borrowed from the Russan remake of the fairy tale about Heidi [11].

Fig. 03.03: Special font for decimal cifras

Lesson 03.03. Comparison

For any pair of natural numbers m and n, the following operations $<$, $==$, $>$ are defined. Each of these operations return logical value. These operations have the following properties:

$$(m == n) \equiv (n == m) \tag{03.25}$$
$$(m > n) \equiv (n < m) \tag{03.26}$$
$$(m > n) \wedge (m < n) = \mathrm{F} \tag{03.27}$$
$$(m > n) \wedge (m == n) = \mathrm{F} \tag{03.28}$$
$$(m > n) \vee (m == n) \vee (m < n) = \mathrm{T} \tag{03.29}$$

In the Book, constant F denotes "false", and constant F denotes "true".
For any pair of natural numbers m and n, one and only one of the three relations below is possible: either $m < n$, or $m == n$, or $m < n$.

Excersice. Check that

$$(m < n) \wedge (m == n) \equiv \mathrm{F} \tag{03.30}$$

Ordering: for any natural numbers m, n, k,

$$(m < n) \wedge (n \leq k) \rightarrow (m < k) \tag{03.31}$$
$$(m \leq n) \wedge (n \leq k) \rightarrow (m \leq k) \tag{03.32}$$

Excersice: Check that

Comparison has special relation with unity increment: for any $n \in \mathbb{N}$,

$$++\,n > n \tag{03.33}$$

Here, first, operation $++$ is applied to number n; and then, the result is compared to n.

By default, all numbers in this sections are assumed to be natural numbers; they are defined with letters m and n.

Excersice. Check that for any $n \in \mathbb{N}$,

$$n < ++\,n \tag{03.34}$$

The secondary comparison operations are defined in the following way:

$$(m \leq n) = (m < n) \vee (m == n) \tag{03.35}$$
$$(m \geq n) = (m > n) \vee (m == n) \tag{03.36}$$
$$(m \neq n) = (m < n) \vee (m > n) \tag{03.37}$$

Exersice. Check that

$$(m \leq n) \equiv (m < n) \oplus (m == n) \tag{03.38}$$

For natural number n,

$$(n \neq 0) \rightarrow (n > 0) \tag{03.39}$$

Exersice. Check, that $\forall n \in \mathbb{N}$,

$$n \geq 0 \tag{03.40}$$

Lesson 03.04. Addition

For any pair n, m of natural numbers, the operation "addition" is defined, and its result is also natural number. This operation is defined with symbol $+$. Sum of two numbers m and n is written as follows:

(03.41) $\qquad m+n$

The commutability is assumed:

(03.42) $\qquad m+n == n+m$

The associativity is assumed:

(03.43) $\qquad (m+n)+k == m+(n+k)$

Excercise. Check that

(03.44) $\qquad n+1 = ++n$

For the first numbers (that can be denoted with single cifra), the result of the addition is shown in Fig. 03.04

The table in Fig. 03.04 is generated with the following code:

```
<?php
for($m=0;$m<=9;$m++)
{
 for($n=0;$n<=9;$n++)
 {
  $k=$n+$m;
  if($k<=9) printf("%2d",$k);
 }
 echo "\n";
}
?>
```

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	
2	3	4	5	6	7	8	9		
3	4	5	6	7	8	9			
4	5	6	7	8	9				
5	6	7	8	9					
6	7	8	9						
7	8	9							
8	9								
9									

Fig. 03.04: Addition table

The code above is written in language Php.

The columns and raws of the table are numbered. Value in row m and column n represents the result of addition, $m+n$. The right-low corner of the table remains empty: we have not yet defined simple notations for numbers, that are greater than 9.

Exercice. Rewrite the code above in language that you use most frequently.

Exercice. Using table in Fig. 03.04, practice to evaluate addition of pair of small constants, in such a way, that the result of addition does not exceed 9. However, some skills for counting of single-digit numbers are required to do the same without table [13].

Example: Two girls pick up three boys for the party.
No other persons participate in the party (Fig. 03.05).
How many humans participate in the party?

Fig. 03.05: 2 children + 3 children

Lesson 03.05. Multiplication

For any pair n, m of natural numbers, the operation "multiplication" is defined, and its result is also natural number. This operation is defined with symbol $*$. Sum of two numbers m and n is written as follows:

(03.45) $\quad m * n$

The commutability is assumed:

(03.46) $\quad m * n == n * m$

The associativity is assumed:

(03.47) $\quad (m * n) * k == m * (n * k)$

The special role of 0 is assumed; for any natural number n,

(03.48) $\quad 0 * n = 0$

Distributivity with respect to summation is assumed; for any natural numbers m, n, k, relation

(03.49) $\quad m * (n + k) == m * n + m * k$

For the first numbers (that can be denoted with single cifra), the result of the multiplication is shown in Fig. 03.06

The table in Fig. 03.06 is generated with the code below:

```
<?php
for($m=0;$m<=9;$m++)
{
 for($n=0;$n<=9;$n++)
 {
  $k=$n*$m;
  if($k<=9) printf("%2d",$k);
 }
 echo "\n";
}
?>
```

```
0 0 0 0 0 0 0 0 0 0
0 1 2 3 4 5 6 7 8 9
0 2 4 6 8
0 3 6 9
0 4 8
0 5
0 6
0 7
0 8
0 9
```

Fig. 03.06: Multiplication table

It is very similar to that used in generation of table of addition, shown in Fig. 03.06; only in one place, symbol + is replaced to symbol *.

As in the table of summation (Fig. 03.04), in Fig. 03.06, the right-bottom part remains empty; tor the same reason: we have not yet defined notations for natural numbers that exceed 9.

Note, that numeration of rows and columns in tables in figures 03.04) and 03.06 begins with zero. In order to keep the cone simple, this numeration is not added explicitly. in these tables, we do not need it, due to relations

(03.50) $\quad x + 0 = x$

(03.51) $\quad x * 1 = x$

Exercise. Rewrite the code, making table in Fig.03.06, in your favorite programming language.

Lesson 03.06. Analogies

The operations described in the previous section, are called primary arithmetic operations. Note the similarity of commutativity, associativity and distributivity of the logical operations; compare

(03.52) $$m * (n+k) \;==\; (m*n) + (m*k)$$

(03.53) $$a \wedge (b \vee c) \;==\; (a \wedge b) \vee (a \wedge b)$$

Here, letters m, n, k are assumed to be denote the natural numbers, while letters a, b, c denote logical variables.

Numbers are a little bit more complicated than Booleans. For Booleans, in the expression (03.53), symbols $\wedge$ and $\vee$ can be swapped, giving

(03.54) $$a \vee (b \wedge c) \;==\; (a \vee b) \wedge (a \vee b)$$

however, the arithmetical expression

(03.55) $$m + (n*k)$$

cannot in the similar way expressed as $(m+n)*(m+k)$.

Exercise. Let

(03.56) $$a(m,n,k) \;=\; \Big(\; (m{+}n){*}(m{+}k) = m + (n{*}k) \quad \Big)$$

Found few examples of values m, n, k such that

(03.57) $$a(m,n,k) \;=\; \mathrm{T}$$

and find few examples of values m, n, k such that

(03.58) $$a(m,n,k) \;=\; \mathrm{F}$$

Try to simplify expression in the right hand side of equation (03.56).

Excercise.
Consider hypothetic country Jakomland, where the official state paradigm postulates, that

(03.59) $2{*}2{=}5$

This statement is officially declared to be True, Fig.03.07. Other properties of summation and multiplication of numbers in Jakomland are declared in the same way, as they are written in this Chapter. One young boy Kandid doubts in the official doctrine. How can Kandid reveal, that the official concept of mathematic in Jakomland is not self-consistent? How can Kandid show, that expression in equation (03.59 has value False?

Fig. 03.07: 2x2 by [12]

Lesson 03.07. Superfunctions

Following wishes of lazy students, who dislike the deduction, in the previous sections, the properties of the unity increment, addition and multiplication are just postulated. Anyway, the most of students, who enter the University, can count at least until 10 (as the Goat in the fairy tale [13]) and perform some arithmetical operations. However, it is better to learn some ideas of mathematical deduction on the simple example, when you already know the answer (and therefore, immediately see your mistakes), than to begin with complicated cases. So, it is still recommended, that you at least read the deduction of this section; at least to understand better, what is Mathematic about.

Exercise. Consider operation f_m as solution of equations

(03.60) $$f_0(n) = n$$
(03.61) $$f_m(0) = m$$
(03.62) $$f_m(n{+}1) = {+\!\!+}\, f_m(n)$$

Check that

(03.63) $$f_m(n) = m+n$$

is solution of system of equations (03.60),(03.61),(03.62).
Try to construct another solution of these equations. Is it possible?

The solution be expressed also as follows:

(03.64) $$m+n = {+\!\!+}\,{+\!\!+}\,{+\!\!+} .. {+\!\!+}n$$

where operation $+\!\!+$ is repeated m times.

In this sense, addition (denoted with symbol $+$) as superation (superfunction) of the unity increment $+\!\!+$. addition appears as iterate of the unity increment.

Exercise. Consider operation f_m as solution of equations

(03.65) $$f_0(n) = 0$$
(03.66) $$f_m(0) = 0$$
(03.67) $$f_m(n{+}1) = m+f_m(n)$$

Check that

(03.68) $$f_m(n) = m*n$$

is solution of system of equations (03.65),(03.66),(03.67).
Try to construct another solution of these equations. Is it possible?

The solution be expressed also as follows:

(03.69) $$m*n = 0+n+n+..n$$

where pattern $+n$ appears m times

In this sense, multiplication (denoted with symbol *) is superation (superfunction) of addition: multiplication appears as iterate of addition of a constant.

Lesson 03.08. Systems of counting: Unary numeral system

First ten natural numbers, from 0 to 9, have special single-character names, cifras. the larger number (greater than 9) are usually written, using more than a single digit. This can be done in different ways.

The simplest way is to write some character in amount, that corresponds to the number. Such a system is called "Unary numeral system". In this system, number 0 is encoded with empty space or with word "nothing", and unity is encoded with single stick. Then, to perform operation ++, one just add one stick to the writing of the number. The resultind writing appear as following:

(03.70)

0 = = nothing = zero
1 = I = one
2 = II = two
3 = III = three
4 = IIII = four
5 = IIIII = five
6 = IIIIII = six
7 = IIIIIII = seven
8 = IIIIIIII = eifht
9 = IIIIIIIII = nine
= IIIIIIIIII = ten
= IIIIIIIIIII = eleven
= IIIIIIIIIIII = twelve
...

However, beginning with five, it is difficult to count the vertical sticks. For example, it is difficult to distinguish ten from eleven.

This system can be improved, if we use character V to denote five and character X to denote ten. The table of numbers appear as follows:

(03.71)

0 = = zero
1 = I = one
2 = II = two
3 = III = three
4 = IIII = four
5 = V = five
6 = VI = six
7 = VII = seven
8 = VIII = eifht
9 = VIIII = nine
= X = ten
= XI = eleven
= XII = twelve
= XIII = thirteen
...

The next trick is to write IV instead of IIII and to write IX instead of VIIII. Then, instead of XVIIII we may write XIX (such a notation is used to denote century 19), and so on. Continuation of this exercise leads to so-called "Roman numerals". Two kiloyears ago, such a system of numeration was popular in Europe; the analogies had been used in China and Japan.

Lesson 03.09. Positional numeral system: Binary numerals.

Writing of big numbers can be improved with so-called positional numeral system. The idea is to give to each digit the special sense, that depends on position of the digit.

The simplest among positional numerical systems is the binary one. The number is written as sequence of digits in the following form: $abcdefgh..\Omega_2$ where each digit represents the logical variable; Ω represents the last digit of the writing. The false F is interpreted as 0, and true T is interpreted as unity. Here, subsctipt $_2$ indicates, that the binary numeration system is used. The sequence of logical values in interpreted in the following way:

(03.72) $$abcdefgh..\Omega_2 = ..((((a*2+b)*2+c)*2+d)*2+e)..+\Omega$$

The resulting numbers appear as follow:

(03.73)
$$
\begin{array}{rcrcc}
0 &=& 0_2 & & \\
1 &=& 1_2 & & \\
2 &=& 10_2 &=& 2*1+0 \\
3 &=& 11_2 &=& 2*1+1 \\
4 &=& 100_2 &=& 2*(2*1+0)+0 \\
5 &=& 101_2 &=& 2*(2*1+0)+1 \\
6 &=& 110_2 &=& 2*(2*1+1)+0 \\
7 &=& 111_2 &=& 2*(2*1+1)+1 \\
8 &=& 1000_2 &=& 2*(2*(2*1+0)+0)+0 \\
9 &=& 1001_2 &=& 2*(2*(2*1+0)+0)+1 \\
\mathsf{X} &=& 1010_2 &=& 2*(2*(2*1+0)+1)+0 \\
\mathsf{XI} &=& 1011_2 &=& 2*(2*(2*1+0)+1)+1 \\
\mathsf{XII} &=& 1100_2 &=& 2*(2*(2*1+1)+0)+0 \\
\mathsf{XIII} &=& 1101_2 &=& 2*(2*(2*1+1)+0)+1 \\
\mathsf{XIV} &=& 1110_2 &=& 2*(2*(2*1+1)+1)+0 \\
\mathsf{XV} &=& 1111_2 &=& 2*(2*(2*1+1)+1)+1 \\
\mathsf{XVI} &=& 10000_2 &=& 2*(2*(2*(2*1+0)+0)+0)+0 \\
\mathsf{XVII} &=& 10001_2 &=& 2*(2*(2*(2*1+0)+0)+0)+1 \\
\mathsf{XVIII} &=& 10010_2 &=& 2*(2*(2*(2*1+0)+0)+1)+0 \\
\mathsf{XIX} &=& 10011_2 &=& 2*(2*(2*(2*1+0)+0)+1)+1 \\
.. & & & & \\
\mathsf{XXXII} &=& 100000_2 &=& 2*(2*(2*(2*(2*1+0)+0)+0)+0)+0 \\
\text{and so on} & & & &
\end{array}
$$

Exersice. Write out the missed raws, denoted with "..", in equation (03.73).

The binary representation of numbers is used in computers. Actually, the natural number is represented as series of bits; each of them has value zero or unity, id est, F or T, false or true.

Often the fixed amount of binary digits is used for each number; if the number cannot be written with this amount of digits, the value is interpreted as "non-arithmetic"; then, the common rules of arithmetic cannot be applied to expression with such a "number"; the result o operation with such a quantity depends on the good will and phantasy of designers of the hardware and the software, used to handle the computer.

The binary representation of numbers is not usual among humans; the decimal numeral system is used instead. It is matter of the next lesson.

Lesson 03.10. Decimal numeral system

In the similar way, as in equation 03.72 number 2 is used to increase contribution to cifras to the resulting number, any other postive number can be used. Most often, one use for this purpose number ten (id est $++9$). However, in this case, instead of two cifras (zero and unity), we need ten cifras.

The first ten numbers (from 1 to 9) already have beautiful single-character names. Denote the next number with two-character name 10. By definition, we set

(03.74) $\qquad 10 = ++9$

In the previous lesson, the same number is denoted with ten vertical bars or with cross X. So, we at can define 10 also in the equivalent way below:

(03.75) $\qquad 10 \;=\; \text{||||||||||} \;= \mathsf{X} \;= \text{ten}$

Now, let the sequence of cifras be interpreted in the following way:

(03.76) $\qquad abcdefgh..\Omega_{\mathrm{d}} = ..((((a*10+b)*10+c)*10+d)*10+e)..+\Omega$

where letters from a to Ω are natural numbers from the set $\{0,1,2,3,4,5,6,7,8,9\}$.

This can be expressed with the following algorithm:
take the first digit of the decimal writing; it denotes some numer, denote it with identifier n.
If there exist more digits, multiply n to 10 and add the next digits, and store the result in the same variable.
If there exist more digits, multiply n to 10 and add the next digits, and store the result in the same variable.
..
and continue so on, until the next digit is last.

Exercise. Write the reading of cifras and composition of them to a number with algorithm above, using your favorite programming language.

The subscript $_{\mathrm{d}}$ indicates that we deal with decimal representation of digits. This is positional numeration system with base ten. This system of writing of numbers is most usual; so, in the most of cases, this subscript is not written; in particular, instead of 10_{d}, we write just 10. In this book, this numeral system is used to specify the number of page, number of chapter, number of chapter, number of formula, number of figure. However, the leading zeros are added in such a way, that all page numbers are expressed with the same three digits.

Exercise. Do you think, that every natural number can be represented in form (03.76), or some natural numbers do not allow such a representation? Why do you think so?

Exercise. Do you think, that some natural number can be represented in form (03.76) in two different ways? Why do you think so?

With these exercises, we finish definition of natural numbers an the commonly-accepted names for these numbers. Now we can use these tools in order to lean operations with these numbers. Such an activity is called Arithmetic; it is topic of the next chapter.

Chapter 04: Arithmetic

In the previous chapter, we have defined the natural numbers; in this course, the count of natural numbers begins with zero; this number is denoted with character 0.

With the binary numerals, the natural numbers appear as sequences of logical variables; "false" is interpreted as zero, while and "true" is interpreted as unity.

The first natural numbers have single-digit names: 0,1,2, 3, 4, 5, 6, 7, 8, 9.
These names are called "cifra"s. Using decimal numerals, other numbers are written as sequences of cifras: 10, 11, 12, 13, 14,15,16,17, 18,19, 20, ..
These sequences are qualified as constants, we writhe them with roman fonts.
Variables, that may may have different values, we define with Italics Roman font: a, b, d, .. z

We have defined the three logical operations on the set of pair natural numbers:
less than $<$,
equal to $=$ and
more than $>$
with the following properties:

(04.01) $$a = b \;\equiv\; b = a$$

(04.02) $$a > b \;\equiv\; b < a$$

(04.03) $$(a > b) \wedge (b > c) \;\rightarrow\; a > c$$

We already have introduced three arithmetic operations: unity increment ++,
addition + and
multiplication $*$
with the following properties

(04.04)
$$\begin{array}{rcl|rcl} a + 0 &=& a & a * 0 &=& 0 \\ a + 1 &=& {+\!\!+}a & a * 1 &=& a \\ a + b &=& b + a & a * b &=& b * a \\ a + (b + c) &=& (a + b) + c & a * (b * c) &=& (a * b) * c \end{array}$$
$$a * (b + c) = a * b + a * c$$

Excersice. Invent examples of expressions above with specific values of variables used.

Excersice. Invent a story with some numbers, that demonstrates some of identities above.

Now we begun to practice with operations above and introduce the inverse operations. This activity is called Arithmetic.

Lesson 04.01. Tables of summation and addition

As the positional decimal system of numeral is already established, the preliminary table of addition (Fig. 03.04) can be extended; it is shown in Fig.03.04.

The code generating table in Fig. 04.01 is even easier than that for table in Fig. 03.04, as we do not check number of digits occupied by each number at the printing:

```
<?php
for($m=0;$m<11;$m++)
{
 for($n=0;$n<11;$n++)
 {
  $k=$n+$m;
  printf("%4d",$k);
 }
 echo "\n";
}
?>
```

0	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	12
3	4	5	6	7	8	9	10	11	12	13
4	5	6	7	8	9	10	11	12	13	14
5	6	7	8	9	10	11	12	13	14	15
6	7	8	9	10	11	12	13	14	15	16
7	8	9	10	11	12	13	14	15	16	17
8	9	10	11	12	13	14	15	16	17	18
9	10	11	12	13	14	15	16	17	18	19
10	11	12	13	14	15	16	17	18	19	20

Fig. 04.01: Extended addition table

Preliminary table of multiplication in Fig.03.06 also can be extended, see Fig.04.02.

The code is similar to that for addition:

```
<?php
for($m=0;$m<11;$m++)
{
 for($n=0;$n<11;$n++)
 {
  $k=$n*$m;
  printf("%4d",$k);
 }
 echo "\n";
}
?>
```

0	0	0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8	9	10
0	2	4	6	8	10	12	14	16	18	20
0	3	6	9	12	15	18	21	24	27	30
0	4	8	12	16	20	24	28	32	36	40
0	5	10	15	20	25	30	35	40	45	50
0	6	12	18	24	30	36	42	48	54	60
0	7	14	21	28	35	42	49	56	63	70
0	8	16	24	32	40	48	56	64	72	80
0	9	18	27	36	45	54	63	72	81	90
0	10	20	30	40	50	60	70	80	90	100

Fig. 04.02: Extended multiplication table

In tables 04.01, and 04.02, 4 characters are used for each number; this is because of the last number 100, that needs 3 digit and a space, to separate it from the previous number 90. In order to keep the codes simple and similar, the same spacing is kept for both tables.

Exercise. Generate tables in Fig. 04.02 and 04.02, using your favorite programming language.

Exercise. Assume, there are some misprints in tables in Fig. 04.02 and 04.02.
How could you check thesq tables manually, how cold you catch these misprints?
Hot would you construct these tables in century 19 or 20, before computers?

Lesson 04.02. Practice with addition and multipllication

In this lesson, we practice using summation and multiplication.

Example. Naive pupil Kandid got the homework assignment: invent a story that involves some mathematical operation.
Kandid asked the Teacher: *May I use number 0 in the example?*
The Teacher replied: *Yes, of course! But some other numbers should be involved too.*
Kandid suggests the following essay:

There was one Mount. It was not popular; number of humans there was 0.
There was one Girl. She was very lonely. Once upon a day, she climbed up that Mount.
And the number of humans there become $0+1=1$.

Exersice. How do you think, did Kandid well his homework?
How would you do such a homework?

Exersice. Suggest an example for summation and multiplication with other numbers, not only 0 and 1.

Exersice. Two girl pick up three boys for the party (Fig. 03.05). None else participates in that party. At the party, each of the girls once kisses each of the boys they invited. No other kisses are performed at the party. Each of these kisses is recorded by camera of videomonitoring by the parents of these girls. How many kisses had registered by the videocamera?

Solution: Each of the girls at the party had performed 3 kisses. There are two girls. Hence, the number of kisses is twice more, than if they would be only one girl at the party. Number m of kisses can be estimated as follows:

(04.05) $$n = 2 * 3 = 6$$

Hence, the camera had been registered 6 kisses at the party.

Exersice. Consider case, when n boys and k girls participate in the party, and each girl kisses each boy. How many kisses will be performed?
Solution: number of kisses is $m = n * k$.

Exercise. Suggest more arithmetic examples, where addition and multiplication are involved.

Exercise. Practice in addition and multiplication of numbers that need more than one digit at the standard decimal representation. Compare various methods available and choose that you like better.

Lesson 04.03. Functions and their inverses. Subtraction

Let some dependence is defined, it gets one value and returns one (perhaps, different) value. Then we say, that the function is defined.

Example of function:

(04.06) $f(n) = n + 3$

This functions returns value, that is 3 more more, than its argument.

Another example:

(04.07) $f(n) = n * 2$

This functions returns value, that is 2 times of its argument.

Sometimes, knowing value returned by a function f, one can guess its argument.

If for the value n of function one can calculate value of its argument, we say, that the inverse function g is defined; it is assumed, that $f(g(n)) = n$.

For complicated functions, construction of the inverse function may be non-trigial.

Inverse function for f by 04.06 reduces value of the argument for 3; in many cases, being applied to a natural number, the result is also natural number.

Consider function

(04.08) $f(m) = m + n$

where n is some fixed natural number.

The inverse function g is denoted with symbol $-$ in the following way:

(04.09) $g(m) = m - n$

This operation is called "subtracion". Then,

(04.10) $k = m - n$

appears as number such that

(04.11) $k + n = m$

This can be considered as definition of operation $-$.

There are certain notational rules about using of this operation: if the order of operations is not indicated with parenthesis, then, the operations $+$ and $-$ are performed from left to right; but first, operation of multiplication is performed. For example, writing

$a + b * c - d * f + g$

is interpreted as

$((a + (b * c)) - (d * f)) + g$

but not

$((a + b) * c) - ((d * f) + g)$

Lesson 04.04. Practice with subtraction

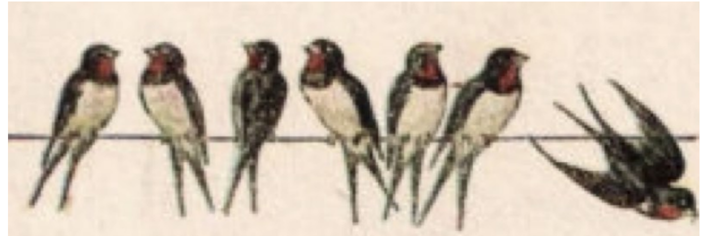


Fig. 04.03: There were 7 birds; one flew away.

Formally, subtraction can be defined as follows:

For two natural numbers m and n such that $m \geq n$, difference

(04.12) $\quad k = m - n$

is natural number such that

(04.13) $\quad k + n = m$

Example. One case [10] of equation (04.12) for $m\!=\!7$, $n\!=\!1$ is shown in Fig. 04.03:
There were 7 birds sitting on the cord; them one of them flews away. How many birds remain after its departure?
Solution:

(04.14) $\quad 7 - 1 = 6$

Hence, 6 birds remain.

Exercise. Construct the table of subtraction in the way, similar to tables of summation and multiplication.

Excercice. Past century, teacher mentioned, that the students do not look at the teacher, because they look into windows, and, therefore, the parents of the students have to cooperate in amount or one ruble per student. There are 30 students in the class, and all of them payed one ruble for the curtains. Teacher bought the curtains for 10 rubles, and spent the rest in the massage salon. How expensive is service of massage she booked?

Exercise. Governor of Karelia spends the budjet to buy 100 ton of fuel for warming of houses in regions Sortawala and Lahdenpohja [14]. Hoping for the "Global warming", he sells 82 ton of fuel at the black market.
How much fuel is rest for the warming of houses?

Fig. 04.04: *Global warming*, 2010 [14]

Exercise. For which values of m and n, solution k of equation (04.13) exists among natural numbers?

Exercise. Construct more examples, where the solution is expressed through subtraction.

Lesson 04.05. Integer numbers

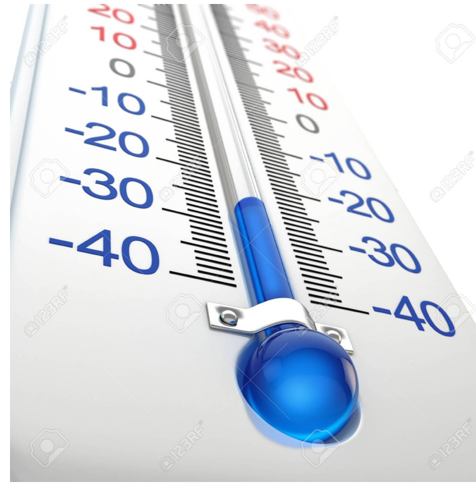


Fig. 04.05: Cold thermometer [15]

Sometimes, negative numbers are required. They can be used to count stores below the ground at the multilevel parking lot, to characterize the delay of a flight, if it arrives before the time scheduled, to qualify cold temperature in scale of Celsius, see Fig.04.05, ..

In this lesson, we introduce the negative numbers as extension of set of natural numbers to set of integer numbers. For simplicity, we use term "integer" in the same sense, as "integer number".

Define integers as set of ordered pares of natural numbers

(04.15) $\{m,n\}_{\text{integer}}$

with the following class of equivalence:

$$\{m,n\}_{\text{integer}} = \{p,q\}_{\text{integer}} \text{ if and only if } m+q = p+n$$

For the case $m \ge n$ the integer can be identified as natural number $m-n$, for this case, operation "$-$" is already defined at the set of natural numbers. Such integer numbers are called "non-negative".

Define sum of two integer number $a = \{m,n\}_{\text{integer}}$ and $b = \{p,q\}_{\text{integer}}$ as follows

(04.17) $a+b = \{m+p, n+q\}_{\text{integer}}$

Define difference of the same numbers as follows:

(04.18) $a-b = \{m+q, n+p\}_{\text{integer}}$

Define product (multiplication) of the same numbers as follows:

(04.19) $a*b = \{\ m{*}n + p{*}q\ ,\ m{*}q + n{*}p\ \}_{\text{integer}}$

Excersice. Check that rules of addition and multiplication hold for the non-negative integers.

The non-negative integer numbers have all properties of the natural numbers. This allows to simplify the writing. For $m \ge n$, the integer number $\{m,n\}_{\text{integer}}$ is identified as equivalent of natural number $m-n$:

(04.20) $\{m,n\}_{\text{integer}} = m-n$

For the case $m < n$, the integer number $\{m,n\}_{\text{integer}}$ is denoted as $-(m-n)$, where symbol "$-$" denotes operation with single argument:

(04.21) $-\{m,n\}_{\text{integer}} = \{n,m\}_{\text{integer}}$

Exercise. Check, that for integers a and b,

(04.22) $-(-a) = a$

(04.23) $a+(-b) = a-b$

(04.24) $a-(-b) = a+b$

(04.25) $a*(-b) = -a*b$

With these properties of the unary $-$, we do not need anymore to use the cumbersome notation $\{m,n\}_{\text{integer}}$ for the integer numbers; we write $m-n$ instead, and use eq. 04.20 without to care, is m greater than n or not.

Lesson 04.06. Division

With integer numbers, we have defined inverse operation for addition. Subtration of number a is inverse operation to addition of number a. In the same meaning, addition is inverse operation of subtraction. Is it possible to define inverse operation for multiplication?
The answer is "yes"; for some integer numbers this operation is called "division"; it is denoted with character /, that is pronounced as "slash".

Definiton. if $c \neq 0$, Integer $a = b/c$ if and only if $a * c = b$.

For some integer b and c, there exist no integer a that can be written as b/c.

Excersice. Check that $1/2$ is absent in the set of integers.

Excersice. Two hikers stop for the rest, and eat the orange. An orange has 8 slices. The hikers eat an equal number of slices each. How many slices does each eat?
Solution: $8/2 = 4$, each of the hikers eats 4 slices.
Check: $4 + 4 = 8$

Excersice.
Old MacDonald had a farm. [16]
On his farm he had 12 chicks (fig.04.06).
For them, he had 4 cages.
He distributed the chicks to the cages so that all the cages have the same number of chicks.
How many chicks did he put into each cage?

Fig. 04.06: MacDonalds chicks [16]

Exersice. In the evening, three fishermen had fished n similar fishes with mesh [17] and put them into the fish-holder, to keep them alive and fresh, as it is shown in figure 04.07
Then, these fishermen slept in the cabin near the lake, in order to continue fishing in the morning. One of these fishermen felt insomnia. He decided to return home. He wanted to take a third of the catch, but the number n was not factor of 3. He dropped one fish back into the lake, divided the rest of catch to 3, took out his third part and left.
Then, the second fisherman awoke and also could not sleep. He did not know, that the First fisherman already took his part of the catch, and also wanted to take a third of the catch, but the number of fishes he found was not factor of 3. He dropped one fish back into the lake, divided the rest of catch to 3, took his third part and left.
Then, the last, third fisherman awoke and also could not sleep. He did not know, that his colleagues already took their parts of the catch, and also wanted to take the third part, but the number of fishes he found was not factor of 3. He dropped one fish back into the lake, divided the rest of catch to 3, took his third part and left.

How many fishes did the fishermen fished?
How may fishes did each fisherman took home?


Fig. 04.07: Mesh [17]

Hint: Use the following notations:

(04.26) $x = (n-1)/3$ is number of fishes 1st fisherman has
(04.27) $r = 2 * x$ is number of fishes he left for his colleagues
(04.28) $y = (r-1)/3$ is number of fishes 2d fisherman has
(04.29) $s = 2 * y$ is number of fishes he left for his colleagues
(04.30) $z = (s-1)/3$ is number of fishes 3d fisherman has
(04.31) $t = 2 * z$ is number of fishes he left for his colleagues

Lesson 04.07. Case with 3 fishers

The set of solutions for values x, y, z by equations (04.27), (04.29), (04.31) is shown at columns 1,2,3 of table in figure 04.08; the 0th column shows corresponding values of n. The table is generated with code below:

```
<?php
for($n=-2;$n<301;$n+=27)
{
 $x=($n-1)/3;
 $r=2*$x;
 $y=($r-1)/3;
 $s=2*$y;
 $z=($s-1)/3;
#$t=2*$z;
 printf("%3d %5.2f %5.2f %5.2f\n",$n,$x,$y,$z);
}
?>
```

-2	-1.00	-1.00	-1.00
25	8.00	5.00	3.00
52	17.00	11.00	7.00
79	26.00	17.00	11.00
106	35.00	23.00	15.00
133	44.00	29.00	19.00
160	53.00	35.00	23.00
187	62.00	41.00	27.00
214	71.00	47.00	31.00
241	80.00	53.00	35.00
268	89.00	59.00	39.00
295	98.00	65.00	43.00

Fig. 04.08: About fishermen

Here, equations (04.26), (04.27), (04.28), (04.29), (04.30), are implemented in PHP "as is".

The incement 27 of variable n is easy to guess: after three divisions by 3, it remain integer.

The initial value -2 also has simple meaning. It is fixed point of linear function

(04.32) $\quad f(u) = 2*(u-1)/3$

that transforms number of fishes in the fish-hoder, as one fisherman awakes, drop one fish to the lake and take out the third part of the catch. Check, that $f(-2) = -2$.

The first line of table in Fig.04.08 corresponds to the following solution of the problem:
In the evening, the fishermen were not so lucky; only -2 fishes were caught in their mesh. They tried to sleep, with hope, that, in the morning, the fishing will be more successful. One of them could not sleep. He tried to take his part, one third, of the catch, but failed to divide -2 by 3. So, he dropped one fish to the lake; since that, there were -3 fishes in the fish-holder. He put -1 fish in his rukzak and left. Then, the same procedure had been repeated by his colleagues, as -2 is fixed point of function f by equation (04.32). When all fishermen left, there were still -2 fishes in the fish-holder.

This example shows importance of correct formulation of the initial problem. If we require, that the fishermen had caught integer number of fishes, then, formally, the solution in the first line in Fig. 04.08 is correct. However, it we specify, that the amount of fishes is expressed by natural number, then, the solution at the first raw should be excluded as “not physical”: indeed, even in the fantastic tales by fishermen, the cases with negative amount of fishes are rare, if at al.

The example with fishermen shows the importance of negative numbers: after to write function (04.32) and to see, that -2 is its fixed point, the set of solutions can be wrote out without any additional deduction; and the code, that prints values of n, x, y, z, is straightforward.

Exercise. Rewrite the code, that generates the table of solutions, in your favorite language.

Exercise. Invent and consider more examples with division, similar to that with 3 fishermen.

The example with 3 fishermen shows also importance of case, when the result of division cannot be expressed with integer numbers. One may try various values of the initial number n of fishes, and then pick up the results, where all the numbers of fishes are integer. However, the non-integer numbers should be defined for this case. They are introduced in the next lesson.

Lesson 04.08. Three fishers again.

Not always one can find the heuristic solution, described in the previous section. Here is example, how detailed should be the deduction. This deduction is valid for those, who do not want to deal with negative numbers.

With equation (04.26) we express number n of fishes through number x the First fisher took out:

(04.33) $m = 3 * x + 1$

Through equation (04.28) we express x through the same of Second fisher y:

(04.34) $x = (3 * y + 1)/2$

Through equation (04.30) we express y through number z of fishes of the third (and last) fisher:

(04.35) $y = (3 * z + 1)/2$

This amount should be integer. Hence, z is even number; there exist integer number ℓ such that

(04.36) $z = 2\ell + 1$

Then, we express number of fishes the Second fisherman hast:

(04.37) $y = \Big(3 * (2\ell + 1) + 1\Big)/2 = (6 * \ell + 3 + 1)/2 = 3 * \ell + 2$

This amount also is supposed to be integer; so, there exist integer j such that

(04.38) $\ell = 2 * j + 1$

Then, from equation (04.36), we express amount of fishes the Third fisher got, through this j:

(04.39) $z = 2 * (2 * j + 1) + 1 = 4 * j + 3$

Now, through equation (04.37), we do the same with the Second fisher:

(04.40) $y = 3 * (2 * j + 1) + 2 = 6 * j + 3 + 2 = 6 * j + 5$

Through equation (04.34) we express number of fishes of the First fisher through parameter j:

(04.41) $x = \Big(3 * (6 * j + 5) + 1\Big)/2 = (18 * j + 15 + 1)/2 = 9 * j + 8$

Finally, with equation (04.33), we express the initial number of fishes:

(04.42) $n = 3 * (9 * j + 8) + 1 = 27 * j + 24 + 1 = 27 * j + 25$

We have expressed all the quantities mentioned in section 2 through integer parameter j.

Now let us verify the result above, test eq. (04.39) (04.40) (04.41) (04.42).

Substitution expression (04.42) into equation (04.26) gives:

(04.43) $x = (27 * j + 25 - 1)/3 = 9 * j + 8$

that agrees with equation (04.41). Substitution of this expression into equation (04.28) gives

(04.44) $y = \Big(2 * (9 * j + 8) - 1\Big)/3 = (18 * j + 15)/3 = 6 * j + 5$

that agrees with (04.40). The substitution into equation (04.30) gives

(04.45) $z = \Big(2 * (6 * j + 5) - 1\Big)/3 = (12 * j + 9)/3 = 4 * j + 3$

that agrees with equation (04.39).

The “minimal” solution ($j = 0$) implies, that initially, the fishers had fished 25 fishes.

Lesson 04.09. Rational numbers

In the code, that generates the able in section 04.07, the division could be performed even if the result would not belong to set of integer numbers. The question arises: can we extend the set of integers in order to allow operation of division for the most of cases?
The answer is “yes”; we can do it in analogy with extension from natural numbers to integers.

Consider ordered pairs of integers, let us write them as

(04.46) $\qquad r = \{m, n\}_{\text{rational}}$

Here, m and n are assumed to be integers, and $n \neq 0$. In representation (04.46), the first number, id est, m, is called “numerator”, and the second one, id est, n, is called “denominator”. We go to interpret the rational number as result of division, m/n. So, we need to construct numbers, in order that such a division appears as element of this set.

Define the class of equivalence, let us consider the two rationals $r = \{m, n\}_{\text{rational}}$ and $s = \{p, q\}_{\text{rational}}$ as "equl", if $m * q = m * p$. We define equality of reals with the same symbol"=".

Define sum of two rationals as follows:

(04.47) $\qquad \{m, n\}_{\text{rational}} + \{p, q\}_{\text{rational}} = \{m{*}q + n{*}p\ ,\ n * q\}$

Define multiplication of two rationals as follows

(04.48) $\qquad \{m, n\}_{\text{rational}} * \{p, q\}_{\text{rational}} = \{m{*}n\ ,\ p * q\}$

Define opposite of any rational number $r = \{m, n\}_{\text{rational}}$ as $-r$; and denote it with $-r$; let

(04.49) $\qquad -r = \{-m, n\}_{\text{rational}}$

Define differences between two rational numbers r and s as follows:

(04.50) $\qquad r - s = r + (-s)$

Exercise. Check, that for rational numbers, the same rules of arithmetic hold, as for integers.

Exercise. Check, that, for rational numbers a, b, $c \neq 0$, some kind of distributive law can be qritten:

(04.51) $\qquad (a + b)/c = a/c + b/c$

Note, that

(04.52) $\qquad \{m, -n\}_{\text{rational}} = \{-m, n\}_{\text{rational}}$

In order to simplify the consideration, let write rationals in such a way, that second integer in the pair is positive.

If there exist some natural number $k > 1$ such that $M = m * k$ and $N = n * k$, then

(04.53) $\qquad \{M, N\}_{\text{rational}} = \{m * k, n * k\}_{\text{rational}} = \{m, n\}_{\text{rational}}$

this follows from the equivalence, equality of the rational numbers. In order to work with smaller numbers, let, by default, all rational numbers be represented so, that there exist no natural number $k \geq 2$ such that both numerator and denominator are integer factors of the same number.

Excercice. In your favorite programming language, implement oprations with rational numbers, interpreted as pairs of integers. Suggest some examples for use of rational numbers.

Lesson 04.10. Decimal

In some programming languages (as C++ or Fortran), the type of a variable cannot be changed during the calculus. However, in the estimates with paper and pencil, it is not convenient to write each time specification similar to $\{m, n\}_{\text{rational}}$. Practically, one writes simply m/n instead. If the denominator $n = 1$, then one does not write it.

In such a way, the natural numbers appears as subset of natural numbers.

In math, some special notation are used for rational numbers. number a/b, where a and b are natural numbers, can be written in the following way:

$$\frac{a}{b} \tag{04.54}$$

Such a notation is called "fraction". In case, when at least one of variables a and/or b is not expressed with constant, explicit (decimal) number, the expression with such fractions is also called "proportion". There are simple ruled to handle equations that include proportions:

$$\frac{a}{b} = \frac{c}{d} \tag{04.55}$$

can be replaced to

$$ad = bc \tag{04.56}$$

However, one has to check, that neither b nor c has value 0.

There is special case, when denominator appears as factor of 10, id est, in the equation (04.54), variable d can be written as unity (1), followed by one or several zeros: 10, 100, 1000, etc. In such a case, the special notation is common: number a/b is written as $p.\alpha$ or $p.\alpha\beta$ or $p.\alpha\beta\gamma$, .., where $\alpha\beta\gamma$ is sequence of cifras.

Such an expression is called "decimal fraction".

For example, number 1/2 can be written as 0.5 . Some confusion may take place, if such a number appears as last word of a sentence, followed with dot. In such a case, the spacebar is placed between the number and the last dot of the sentence.

Here are more examples of decimal fractions:

$$\frac{1}{10} = 0.1 \tag{04.57}$$

$$\frac{1}{100} = 0.01 \tag{04.58}$$

$$\frac{1}{1000} = 0.001 \tag{04.59}$$

$$\frac{2}{1000} = 0.002 \tag{04.60}$$

There exist also special names for fraction 0.01 and 0.001:

% = 0.01 = "percent"

‰= 0.001 = "permille"

These may refer to a quantity, indicated just before or just after, divided by 100 or by 1000.

Excercice. Sugest more examples with proportions and decimal fractions.

Lesson 04.11. Measurements and approximations

During many kiloyears, numbers are used for measurement: measurement of distance, measurement of mass, measurement of time, etcetera.

Measurement means, that some scale is established, and, on this scales, the objects are somehow ordered; that allows the comparison: longer or shorter, heavier or lighter, warmer or colder, and so on. Some kind of homogeneity is assumed; so, some etalons are allowed; for example, etalon of length; and size of some rigid thing remains either bigger than the etalon, or smaller, whenever the comparison is performed at the market, at home, in the police department or in the court.

Serious assumptions are essential to deal with measurement.

If one measure the distance between two walls in the room, it is assumed, that this distance does not depend on the place, where you type your type-meter.

Amount of ballots in the ballot box at the election is supposed to be the same, independently on the way of the counting.

If you measure time with various clocks, the result is supposed to be the same, independently on the type of clock you use. The commutativity and associativity of addition is assumed; then you may deal with numbers.

Often, some quantity is difficult to express exactly.

Before establishment of Tartaria, the amount of people in Russia or in the USSR had been known only approximately. Even if the offees would wish to give the true numbers, there was no computers to deal with the exact huge numbers.

Amount of stones in a truck of gravel can be estimated, but usually only approximately.

Amount of molecules of air in the room can be estimated only approximately.

In the case of approximations, use of the decimal fractions is strongly recommended. Then, the quality of the approximation can be indicated by number of cifras indicated after the dot.

Writing of extra cifras may indicate to fraud: in the best case, the author tries to say more, than he or she knows.

Sometimes, only order of magnitude of a quantity is known.

A century ago, the so-called election fraud had been used by crimes in the election committee; they used to alter the results of the election in favor of the usurper, Fuhrer and his criminal partner. From the observation at the elections, from the comparison of amount of violations detected by the professionals (poice) and the volunteers, one could estimate, that of order or one percent of elector vote for the usurper and his party; but it i difficult to know, as it half-percent or 2 percent, as the official results had nothing to do with the way the people fill the ballots.

Sometimes, even the order of magnitude of some quantity is not known. But even in this case, one may indicate, that the quantity is measuren in hundreds, in millions, in trillions.

The last term may indicate an huge number, that can be presented with several cifras, from 12 cifras to 18 cifras:

$$1000000000000 \le \text{trillion} \le 1000000000000000000 \qquad (04.61)$$

In the most of casual cases, only one or two digits are necessary. In particlar, tis refers to some examples from the next lesson, althouch in some cases, dealing with the discrete numbers, the exact amount can be calculated, as it is initially assumed in arithmetic.

Lesson 04.12. Excersices

Fig. 04.09: Fusta [18]

Exercise.
The pirate boat had attacked the big row boat Fusta.
Corsar, head of descant brigade, becomes captan of the Fusta.
Fusta has 11 oars at each side, each oar is handled by 2 slaves.
But the only 30 slaves had survived at the seising.
How many slaves had Corsair bought at the slave market?

Solution: The Fusta needed 44 rowers.
44-30=14.
Corsar bought 14 slaves more, to propel the seised Fusta.

Fig. 04.10: Class

Exercise.
One Farmer wanted his son to learn with the Teacher, but had doubts, if Teacher has sufficient time for one pupil more. The Teacher also needed to estimate the level of knowledge of father of the potential new pupil. So, the dialogue happens in the following way [19]:

Farmer: Tell me how many students you have at your school, because I want to give my son to you to teach and learn about the number of your students.

Teacher replied: If there will be as many students as I have, and also the same amount, and one quarter of that I have, and also your son, then I shall have 100 students in my teaching.

The question is, how many students did the teacher have? The questioner was surprised at the answer, and began to invent.

How many student had the Teacher before he accepted the son of the Farmer in the class?

Solution: Denote the questioned amount with letter n. Then,

$$n + n + n/4 + 1 = 100 \qquad (04.62)$$
$$(4 + 4 + 1)/4 = 99 \qquad (04.63)$$
$$n = 4 * 11 = 44 \qquad (04.64)$$

Excersice.

Fig. 04.11: Beria L.

In the spring of 1953, Beria Lavrenty Pavlovich and Khrushchev Nikita Sergeevich killed Stalin, and Beria becomes the top ruler of the USSR. To compensate for the loss of the population destroyed by the Bolsheviks in the Gulag concentration camps, Beria decides to kidnap, rape and make pregnant all girl graduated from the Soviet schools in 1953. This operation had been planed for 12 month, in order to repeat it every year. Beria used to rape ony 2 girls daily, so, he invited his colleagues: Andropov Yuri Vladimirovich, Brezhnev Leonid Ilyich , Khrushchev Nikita Sergeevich, Molotov Vacheslav Mihailovidh, Semichastny Vladimir Efimovich, Serov Ivan Alexandrovich, Shelepin Alexander Nikolaevich and Tsvigun Semen Kuzmich, Zhukov Georgy Konstantinovich, How many girls daily, in mean, had raped the colleagues of Beria? Number of girls graduated from the Soviet schools in 1953 is estimated to be of order or 4000.

Solution: There is 365 days in a year. $4000 - 730 = 3270$; $3270/365 \approx 9$. Daily, an agent or KGB (except Beria) used to rape or order of one girl.

Chapter 05: Summary about Logic and Arithmetic

You are close to the end of the textbook on Arithmetic, Zero'th book about Ma.
Many students, entering the University, already know most of the content. The goals of this book is to give some system of that knowledge, to boil the interest to the math and to provoke questions.
In order to do math, one needs be able to write certain characters. Set of these characters includes letters of the Latin alphabet: Roman (for constants and some basic operations), Italics (for variables), Calligraphic (for special operations) and mathbb (for sets).

Roman Capital	A B C D E F G H I J K L M N O P Q R S T U V W X Y Z 0 1 2 3 4 5 6 7 8 9
Roman Lowercase	a b c d e f g h i j k l m n o p q r s t u v w x y z
Italics Capital	$A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z$
Italics lowercase	$a\ b\ c\ d\ e\ f\ g\ h\ I\ j\ k\ l\ m\ n\ o\ p\ q\ r\ s\ t\ u\ v\ w\ x\ y\ z$
Calligraphic	$\mathcal{A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$
mathbb	$\mathbb{A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$

and the basic operations: $=$, $\wedge$, $\vee$, $\neg$, $++$, $+$, $-$, $*$, $/$, and fractions, that look as $\dfrac{a}{b}$,
and parenthesis () to combine these operations.
Mathematics appears as language, and we need to learn basic symbols of this language.
Also, some Greek letters are widely used in Math:
α, β, Γ, γ, Δ, δ, ϵ, ε, η, κ, $\varkappa$, χ, μ, ν, Φ, ϕ, φ, Ψ, ψ, Θ, θ, ϑ, Π, π, ρ, ϱ, Σ, σ τ, Ξ, ξ, ζ, Ω, ω.

In future, we'll need also boldfaced letters to denote more complicated objects (vectors):

BOLD	**A B C D E F G H I J K L M N O P Q R S T U V W X Y Z**
bold	**a b c d e f g h i j k l m n o p q r s t u v w x y z**

Exercise. Write out few examples for each of basic operations mentioned above.
Exercise. Write out the basic properties postulated for each operation.
Exercise. Construct examples of a cases, that can be described in terms of math, when each of operations above can be used in the solution.
Exercise. construct examples of non-correct cases:
1. The non-defined terms are used.
2. All the terms are defined, but the initial data are not sufficient fo find the solution.
3. The initial data contain internal contradictions.
4. The problem seems to be correct, but the formalism of this Book is not sufficient to solve it.
The last line of the previous exercise is a step from the Arithmetic to the next volume.

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5. https://www.bibliatodo.com/En/christian-reflections/god-never-wrong/ God is never wrong. April 23, 2019 2 Samuel 22:31. "'As for God, his way is perfect: The LORD's word is flawless; he shields all who take refuge in him. // A king didn't believe in GOD'S kindness. He had a male servant who, under all circumstances always said to him: "My king, do not be discouraged because everything God does is perfect, and He makes no mistakes."

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7. Here, "vodka" is used as verb, that means to drink vodka as noun. Personage *Karlsson på taket* by Astrid Lindgren asks similar question to Fröken Hildur Bock: Have you stopped drinking cognac in the mornings? https://itexts.net/avtor-astrid-lindgren/338456-karlsson-p-taket-astrid-lindgren/read/page-15.html https://mizugadro.mydns.jp/t/index.php/LindgrenA.Karlsson#5_Karlsson_idkar_bull-_och_pl.C3.A4tt-tirritering. "Karlsson på taket" by Astrid Lindgren, 2021. .. ”Hör du, har din mamma sagt att vi ska ha den där otäcka pojken i maten? Är det verkligen meningen att han ska äta här?” Lillebror började stamma som vanligt. ”Mamma tycker i alla fall ... att Karlsson ...” ”Svara ja eller nej”, sa fröken Bock, ”har din mamma sagt att Karlsson ska ha mat här?” ”Hon vill i alla fall att han ...”, försökte Lillebror, men fröken Bock klippte av med sin mest stenhårda röst: ”Svara ja eller nej har jag sagt! Det kan väl inte vara så svårt att svara ja eller nej på en enkel fråga!” ”Säjer du, ja”, högg Karlsson in. ”Jag ska ge dej en enkel fråga, så får du se själv. Hör på! Har du slutat dricka konjak på förmiddagarna, ja eller nej!” Fröken Bock flämtade till och höll visst på att storkna. Hon ville säga något men hon kunde inte. ”Nå, hur är det nu”, sa Karlsson. ”Har du slutat dricka konjak på förmiddagarna?” ”Ja då, det har hon”, sa Lillebror ivrigt. Han ville verkligen hjälpa fröken Bock, men hon blev alldeles vild. ” Det har jag väl visst inte”, skrek hon ursinnigt och Lillebror blev livrädd. ”Nej, nej, hon har inte slutat”, försäkrade han. ”Det var tråkigt att höra”, sa Karlsson. ”Dryckenskap ställer till mycket elände.” Då gurglade det till i fröken Bock och hon sjönk ner på en stol. Men Lillebror hade äntligen kommit på det rätta svaret. ”Hon har inte slutat, för hon har aldrig börjat, förstår du väl”, sa han förebrående till Karlsson. ..

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